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$\frac{22}{7}$ AND ALL THAT

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We all know and have used the approximation $\pi \simeq \frac{22}{7}$. It may have occurred to you to ask why this is a worthwhile value. Why not just use $\pi \simeq \frac{31}{10}$ (that is, 3·1), 3·14, 3·142, or ... ?

In a previous article I showed how to use the technique of continued fractions to find the "best" rational approximations to $\log 2/\log \frac{3}{2}$, and hence plausibly to answer the question, "Why are there just twelve semitones in an octave, and not (say) eleven or thirteen?" Here we'll look a bit further into the approximation properties of continued fractions, illustrating these by showing why $\frac{22}{7}$ is in fact a very useful estimate for π. I won't repeat any of the introductory material on continued fractions, so if you need further explanation of this please refer to my earlier article.

In contrast to $\log 2/\log \frac{3}{2}$, where we used a neat (though still fairly laborious) method of finding the partial quotients, the only way I know to find the continued fraction of π is by "brute force": take a decimal approximation, say

$$\pi = 3{\cdot}14159\,26535\cdots$$

and work with this instead of the exact value. Naturally, the more decimal digits of π we have, the further we will be able to calculate the continued fraction. Clearly $[\pi] = 3$, and so the computation begins

$$\pi = 3 + (\pi - 3) = 3 + \frac{1}{1/(\pi - 3)}\,.$$

By calculator, $1/(\pi - 3) = 7{\cdot}06251\,33059\cdots$ and so

$$\begin{aligned}
\pi &= 3 + \cfrac{1}{7 + \left(\cfrac{1}{\pi - 3} - 7\right)} \\
&= 3 + \cfrac{1}{7 + \cfrac{22 - 7\pi}{\pi - 3}} \qquad \text{(do you notice something already?)} \\
&= 3 + \cfrac{1}{7 + \cfrac{1}{(\pi - 3)/(22 - 7\pi)}} \\
&= \cdots \\
&= 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{1 + \cdots}}}}}
\end{aligned}$$

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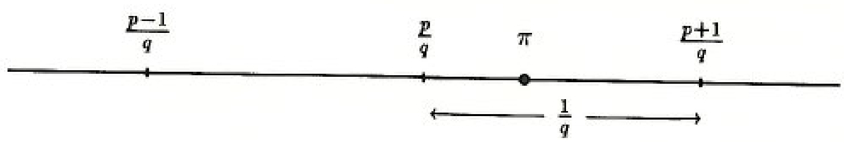
after a long calculation. (The result of an even longer calculation is given in problem 2 at the end in case you would like to play around with it.) So by constructing a table

		3	7	15	1	292	1	...
0	1	3	22	333	355	103993	104348	...
1	0	1	7	106	113	33102	33215	...

we find the first six convergents to π. The fraction $\frac{22}{7}$ duly makes an appearance, and as I wrote in my earlier article, the convergents to a number are (in a sense) the "best" rational approximations to that number. This shows why $\frac{22}{7}$ is, for many purposes, a more useful approximation than, say, $\frac{31}{10}$.

Just how good are convergents as approximations?

Suppose we pick a denominator q at random, and then determine p such that $\frac{p}{q}$ is as close to π as possible. For example, if $q = 10$ we will find $p = 31$, since π is between $\frac{31}{10}$ and $\frac{32}{10}$, and closer to the former.* From the diagram



it is clear that the maximum possible discrepancy between $\frac{p}{q}$ and π is $|\pi - \frac{p}{q}| = \frac{1}{2q}$. If we assume that π is in some sense "randomly placed" among the rational numbers, we could regard the "average" or "expected" discrepancy as being half of this, that is, $|\pi - \frac{p}{q}| = \frac{1}{4q}$. However for convergents, it is possible to show that

$$\left|\pi - \frac{p_n}{q_n}\right| < \frac{1}{a_{n+1}q_n^2}\,, \qquad (*)$$

where a_{n+1} is the next partial quotient in the continued fraction expansion of π. Let's look at a few examples of approximating π by fractions with given denominator.

(1) If we take $\frac{p}{q} = \frac{31}{10}$, the best approximation to π with denominator 10, we find the error to be

$$\left|\pi - \frac{31}{10}\right| = 0{\cdot}04159\cdots \simeq \frac{1}{24}\,,$$

which is roughly the same as (in fact a bit worse than) the "expected" error $\frac{1}{40}$. Note that $\frac{31}{10}$ is not a convergent to π.

(2) Choosing the denominator $q = 7$ and the best possible p, we would "expect"

$$\left|\pi - \frac{p}{7}\right| \simeq \frac{1}{4\times 7} = \frac{1}{28}\,;$$

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* Of course π is merely a familiar example: the same ideas apply in approximating any given number.

however, since $\frac{22}{7}$ is a convergent to π, $(*)$ tells us that

$$\left|\pi - \frac{22}{7}\right| < \frac{1}{15 \times 7^2} = \frac{1}{735} .$$

Thus as an approximation to π, $\frac{22}{7}$ has less than $\frac{28}{735} = \frac{1}{26}$ of the error that we could "reasonably expect"!! To see this in another way,

$$\frac{22}{7} = 3{\cdot}14285 \cdots = \pi \quad \text{to 2 decimal places.}$$

So $\frac{22}{7}$ is just as close an approximation as $\frac{314}{100}$.

(3) We obtain an even more spectacular example by choosing $q = 113$. Then the "average" error is

$$\left|\pi - \frac{p}{113}\right| \simeq \frac{1}{4 \times 113} = \frac{1}{452}$$

if p is chosen as well as possible; but in fact

$$\left|\pi - \frac{355}{113}\right| < \frac{1}{292 \times 113^2} = \frac{1}{37\,28548} .$$

Thus the approximation

$$\pi \simeq \frac{355}{113} = 3{\cdot}14159\,29203 \cdots$$

is accurate to six decimal places; the error is less than $\frac{1}{8200}$ of that expected!!! It is a better approximation than $\frac{3141593}{1000000}$. (The estimate $\pi \simeq \frac{355}{113}$ was known to the Chinese mathematician Zu Chongzhi in the fifth century A.D.)

It is clear from $(*)$ that convergents are good approximations to a number. Indeed, we have

$$\begin{aligned} \frac{\text{true error}}{\text{"expected" error}} &= \frac{|\pi - p_n/q_n|}{1/4q_n} \\ &= 4q_n\left|\pi - \frac{p_n}{q_n}\right| \\ &< \frac{4}{a_{n+1}q_n} . \end{aligned}$$

Since the denominators q_n are always increasing, this ratio becomes smaller and smaller as we move along the sequence of convergents. Moreover, a convergent $\frac{p_n}{q_n}$ will be not only better than "average", but a really exceptional approximation, when the next partial quotient a_{n+1} is large. This was the case in the examples I chose for (2) and (3) above, where a_{n+1} was 15 and 292 respectively.

Convergents are known to be good approximations to a number. However, it might be that there are other fractions (not convergents) which are also good approximations. It turns out that this is not the case – in other words, convergents are (in a sense) the *only possible* good approximations! Consider the relation found above,

$$\frac{\text{true error}}{\text{"expected" error}} < 4q_n\left|\pi - \frac{p_n}{q_n}\right| ,$$

when $\frac{p_n}{q_n}$ is a convergent to π. Now determine the best rational approximations $\pi \simeq \frac{p}{q}$ for $q = q_n + 1, q_n + 2, \ldots$. How long does it take before the ratio of true to average error is better than for $\frac{p_n}{q_n}$? That is, what is the smallest q such that

$$4q\left|\pi - \frac{p}{q}\right| < 4q_n\left|\pi - \frac{p_n}{q_n}\right|$$

if p is chosen as well as possible? It can be shown that this does not happen until $q = q_{n+1}$. In other words, to get a better approximation than some convergent we have to go at least as far as the next convergent. To take a specific example, the fractions

$$\frac{25}{8}, \frac{28}{9}, \frac{31}{10}, \ldots, \frac{327}{104}, \frac{330}{105}$$

are no better approximations to π than $\frac{22}{7}$.

As a final illustration we have tabulated for $q = 1, 2, \ldots, 10$ the closest approximation $\frac{p}{q}$ to π; the error in this approximation; and the ratio of actual error to "expected" error. It can be seen that this ratio is usually fairly close to 1 (exercise: show that it can never be more than 2), but takes a sudden plunge at the convergent $\frac{22}{7}$.

q	$\frac{p}{q}$	error	$\frac{\text{true error}}{\text{"expected" error}}$
1	$\frac{3}{1}$	0·14	0·6
2	$\frac{6}{2}$	0·14	1·1
3	$\frac{9}{3}$	0·14	1·7
4	$\frac{13}{4}$	0·11	1·8
5	$\frac{16}{5}$	0·058	1·2
6	$\frac{19}{6}$	0·025	0·6
7	$\frac{22}{7}$	0·0013	0·04
8	$\frac{25}{8}$	0·017	0·5
9	$\frac{28}{9}$	0·030	1·1
10	$\frac{31}{10}$	0·042	1·7

Problems

(1) Use the value $\pi^2 \simeq 9{\cdot}86960\,44011$ (accurate to ten decimal places) to show that π^2 has a rational approximation with denominator less than 30 and error less than $\frac{1}{2}\%$ of the "average" error. (I estimate that if your calculator works with ten significant digits you will be able to calculate eleven partial quotients accurately, but the twelfth will be wrong. Unfortunately there will be no particular indication that anything is amiss – this is a problem which can only be resolved with some care.)

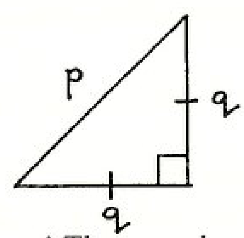
(2) Going beyond what was given above, the partial quotients of π begin

$$3, 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, 2, 2, 1, 84, 2, \ldots\ .$$

Use these to find some more exceptionally good approximations to π.

(3) Construct a table like that above to see what happens near some later convergents. Take, for example, $q = 100, 101, \ldots, 120$.

Another interesting problem is to find a right-angled isosceles triangle with all side lengths integers.



If we have such a triangle, Pythagoras' Theorem gives us $p^2 = q^2 + q^2 = 2q^2$ and hence

$$\frac{p}{q} = \sqrt{2}\,.$$

However this is impossible. (If you have never seen it proved, look up *Parabola* vol. 25, no. 2, p. 4 (1989).) So as an alternative problem we could seek right-angled isosceles triangles with sides as near as possible to integers. This will lead us to find rational approximations to $\sqrt{2}$, and as we already know, this is the sort of problem which continued fractions deal with very well.

First we find the continued fraction of $\sqrt{2}$. This can be done by first finding a decimal approximation, or with more insight as follows. You have probably learned how to rationalise the denominator of a fraction involving surds; recall that in finding continued fractions we wish to rationalise the numerator instead. (In fact we want all the numerators to be 1.) Thus we compute $[\sqrt{2}] = 1$ and then

$$\begin{aligned}\sqrt{2} &= 1 + (\sqrt{2} - 1) \\ &= 1 + \frac{(\sqrt{2}-1)(\sqrt{2}+1)}{\sqrt{2}+1} \\ &= 1 + \frac{1}{\sqrt{2}+1} \\ &= 1 + \frac{1}{2 + (\sqrt{2}-1)}\ .\end{aligned}$$

Observe that the most recent remainder, $\sqrt{2}-1$, is the same as the previous one. Therefore from this point the process repeats over and over again:

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}}}$$

and we can find the first few convergents to $\sqrt{2}$:

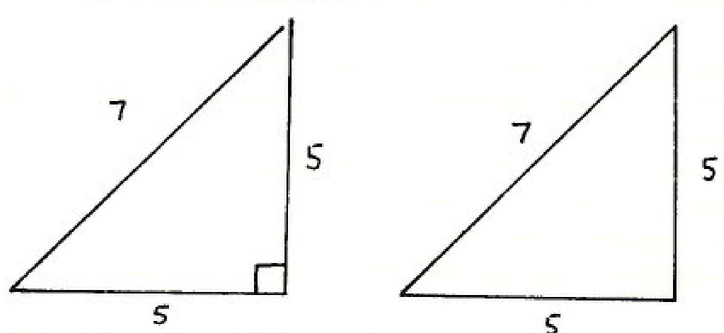
		1	2	2	2	2	2	...
0	1	1	3	7	17	41	99	...
1	0	1	2	5	12	29	70	...

Thus our first approximation will be $\sqrt{2} \simeq \frac{3}{2}$, giving the following attempts at a right-angled isosceles triangle. (If we insist on keeping the sides $2, 2, 3$ we can either make the figure not right-angled, or not a triangle!)

The correct hypotenuse length for sides of length 2 is

$$\sqrt{2q^2} = \sqrt{8} = \sqrt{9} - \text{a little bit} = 3 - \text{a little bit}$$

(compare this with the left half of the above diagram). The next convergent to $\sqrt{2}$ is $\frac{7}{5}$, giving the approximate right-angled triangles



in which the correct hypotenuse should be

$$\sqrt{2q^2} = \sqrt{50} = \sqrt{49} + \text{a little bit} = 7 + \text{a little bit} .$$

And so on...

Problems

(4) Using the value $\sqrt{2} = 1{\cdot}41421\,35628\cdots$ (accurate to ten decimal places), compare the actual error with the "expected" error for some convergents and some non-convergents to $\sqrt{2}$. While the convergents are better approximations than the others, none of them is as exceptional an approximation as, for example, $\pi \simeq \frac{355}{113}$. Why not?

(5) Show that if the shorter sides of a right-angled triangle are integers in the ratio of $2:1$, then the hypotenuse cannot be an integer. Find some triangles of this shape in which the hypotenuse is very nearly an integer.

(6) Find equilateral triangles in which the sides are integers and the altitudes are close to integers.

(7) Show that if the sides of a cube are integers then the main diagonals (that is, the diagonals passing through the centre of the cube) are not. Find some integral side lengths such that the diagonals are very nearly integers. Can you find out anything about the possibility of making not only the sides and the main diagonals but also the face diagonals (that is, the lines from corner to corner of each square face of the cube) close to integers?

Further reading

N.M. Beskin: **Fascinating fractions.** Mir Publishers, 1986.

C.D. Olds: **Continued fractions**. Random House, 1963.

I. Niven and H.S. Zuckerman: **An introduction to the theory of numbers.** Wiley, 1972. (*more technical*)

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"When I considered what people generally want in calculating, I found that it always is a number.

I also observed that every number is composed of units, and tat any number may be divided into units.

I observed that the numbers which are required in calculating by Completion and Reduction are of three kinds, namely, roots, squares and simple numbers relative to neither root nor square.

A root is any quantity which is to be multiplied by itself, consisting of units, or numbers ascending, or fractions descending.

A square is the whole amount of the root multiplied by itself.

A simple number is any number which may be pronounced without reference to root or square.

A number belonging to one of these three classes may be equal to a number of another class; you may say, for instance, 'squares are equal to roots', or 'squares are equal to numbers', or 'roots are equal to numbers'."

from Al-Khwarizmi, **The algebra of Mohammed Ben Musa**, ed. and translated F. Rosen, John Murray; quoted in **The history of mathematics – a reader**, ed. John Faurel and Jeremy Gray Macmillan Education.