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THE DEMONSTRATION OF PYTHAGORAS

By Simon Prokhovnik*

The geometrical proposition, named after the Greek mathematician and philosopher, Pythagoras, ($\sim 570 - 500$ BC), deals with a unique property of right-angled triangles.

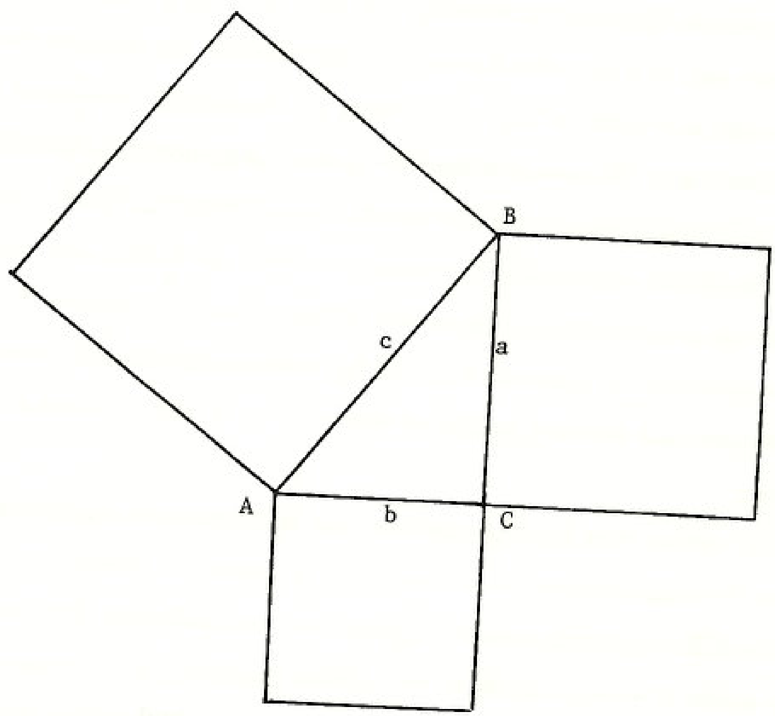


Figure 1

Thus if ABC is a right-angled triangle (as shown), then the area of the square based on the hypotenuse (the side opposite the right-angle) is equal to the sum of the areas of the squares based on the other two sides. Or, more simply, if $\angle A$ is a right-angle, then $a^2 = b^2 + c^2$. There are many ways to demonstrate this result. Probably the longest and most tortuous 'proof' of this 'theorem' is the best-known one since it is presented as one of the theorems in the system of geometry known as Euclid's **Elements**. However, few people remember the outline or point of this proof — which is a pity, since the result is

* Simon is an applied mathematician at the University of New South Wales

one of the most important in mathematics, and spills over as an important feature of our understanding of the universe and of the laws of physics.

Note that I am reluctant about the word 'proof' and prefer 'demonstration' (or possibly deduction) of a mathematical proposition. The point is that any such proposition is ultimately based on assumptions and definitions, unprovable or even arbitrary; and the validity of the proposition depends on whether (or not) you are prepared to accept the assumptions (at least, say, as a basis for discussion) and whether the deductions you make from your assumptions are valid or 'logical' — that is, essentially, consistent with your assumptions and definitions. The merit of Euclid's system of geometry is that his assumptions (his axioms) and definitions are clearly admitted and presented at the start.

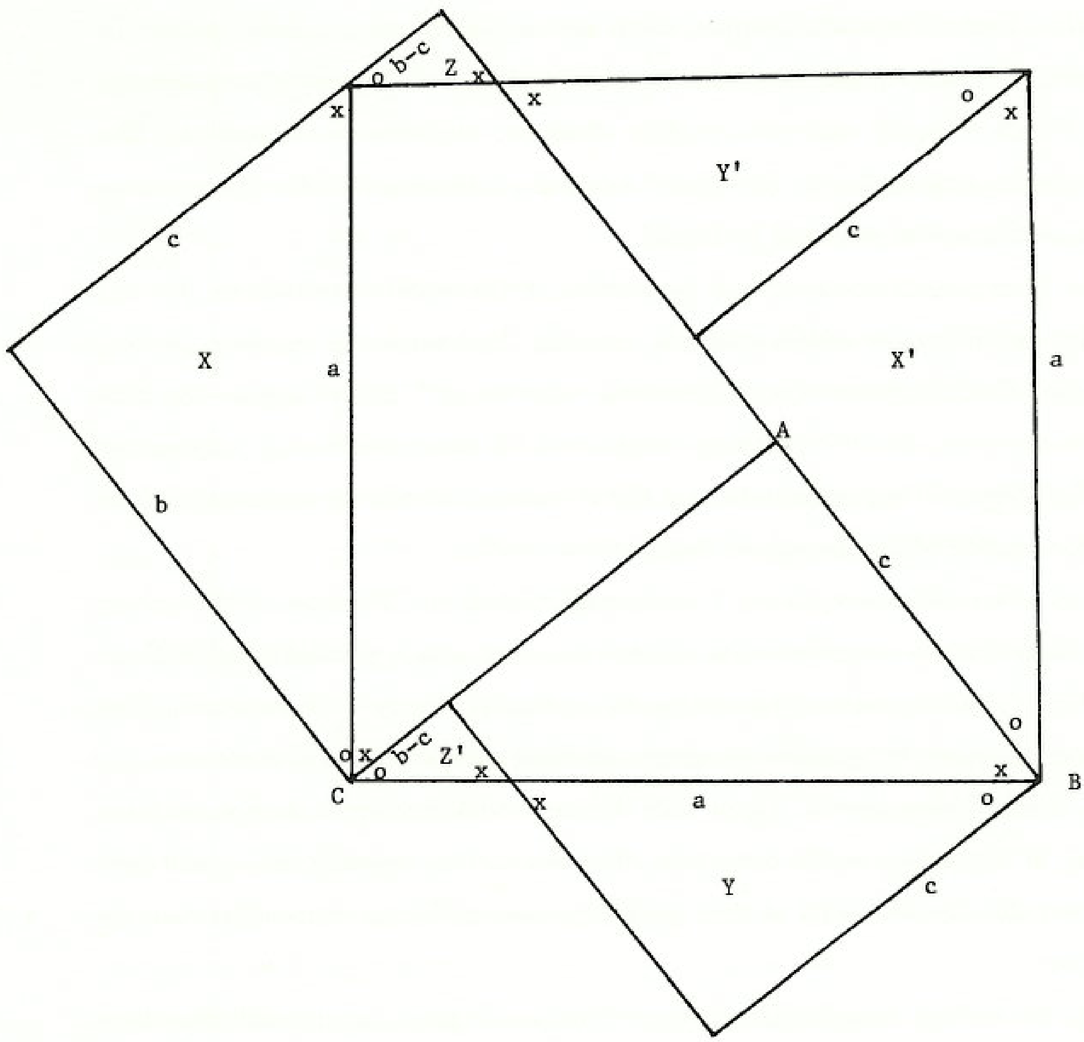
So, quite apart from all the remarkable propositions which, as Euclid was able to show, follow from his axioms (which include arguments of 'logic') and definitions, the structure of his 'system' has provided an important model for mathematicians and scientists. And for philosophers it provided starting points on the notions of truth, physical reality and their representation.

Euclid's systematic outline has for millenia been an inspiration to mathematicians and provided the basis for mathematical education. But, in presenting Euclid's system we still tend to employ the difficult anachronistic language and methods that Euclid employed at a time when the modern tools of algebra, co-ordinate geometry and vector algebra were completely unknown. So, unfortunately, the manner of bringing Euclid to students is in a language much more difficult than it need be, and this often results in a lack of appreciation by students of the beauty of the propositions and their inter-relationship. The Pythagoras theorem is a case in point, and also an example of the different ways by which one can demonstrate a result.

1. A pictorial demonstration

It can be shown in a number of ways that the areas of the squares, based on the sides emanating from the right-angle, exactly fit into the area of the square based on the hypotenuse, for example, as in Figure 2.

Figure 2



It is seen that the sections of the areas X, Y and Z of the smaller squares, which fall outside the big square (based on the hypotenuse), fit neatly into the section X', Y' and Z' respectively, thus completely and precisely filling up the big square. Of course, to appreciate the perfect fit of this result, we need to invoke the notion of similar triangles (which have the same shape), and of congruent triangles (which are of the same size as as well as the same shape); also the meaning of a right-angle, of complementary angles such as out o and x which add up to one right-angle; and the interesting proposition that the interior angles of a triangle add up to two right-angles. All these results are defined or developed in Euclid's system; however, these also are easily demonstrated pictorially (or otherwise). More important, all of these notions and their 'Pythagoras' consequence, depicted in Figure 2, apply only on a surface which we can consider as perfectly flat. Euclidean geometry only applies to "flat space" in which parallel lines behave in accordance with a fundamental axiom proposed by Euclid.

It is of more than passing interest that the surface of the world on which we live and operate is not flat; it is more-or-less spherical, ignoring the bumps and wrinkles. So does that mean that Euclid's system has no practical value for us? By no means: for most purposes (an exception would be airways navigation) his assumptions and propositions apply to a high degree of approximation, and this is usually the best we can hope for from a mathematical model which, hopefully, describes our world.

Of course, we now realise since Gauss, Lobachewski, Bolyai and Riemann (19th century mathematicians) that the employment of assumptions (say, about parallel lines) different to those of Euclid, leads to various types of 'non-Euclidian' geometry — airways navigation requires the employment of spherical geometry. However, our ancestors of 3000 years ago – in China, India, Mesopotamia, Egypt and Greece – who discovered and worked out the geometry of flat space, would have been oblivious to this problem, and could have contended that the obvious utility of their geometry confirmed their supposition that the Earth was flat!

Undoubtedly the earliest demonstrations of the Pythagoras proposition would have been by measurement and pictorial argument. Bronowski in his **Ascent of Man** (his book and T.V. series) imagines how an Egyptian boy, sketching shapes in the sand, might have demonstrated that the squares on the smaller sides of a right-angled triangle fit precisely into the square on the largest side – possibly in quite a different way to the demonstration in Figure 2.

You may be interested to demonstrate the implications of Figure 2 by making a five-piece jig-saw puzzle from it: a puzzle from which two pieces form one square, three pieces another square, and all five pieces the square on the hypotenuse.

2. A demonstration using symmetry and algebra Our Egyptian boy, tracing triangles in the sand, may well have come across the result that if the triangles are right-angled and of the same size four of them fit neatly together to form a square, as in Figure 3.

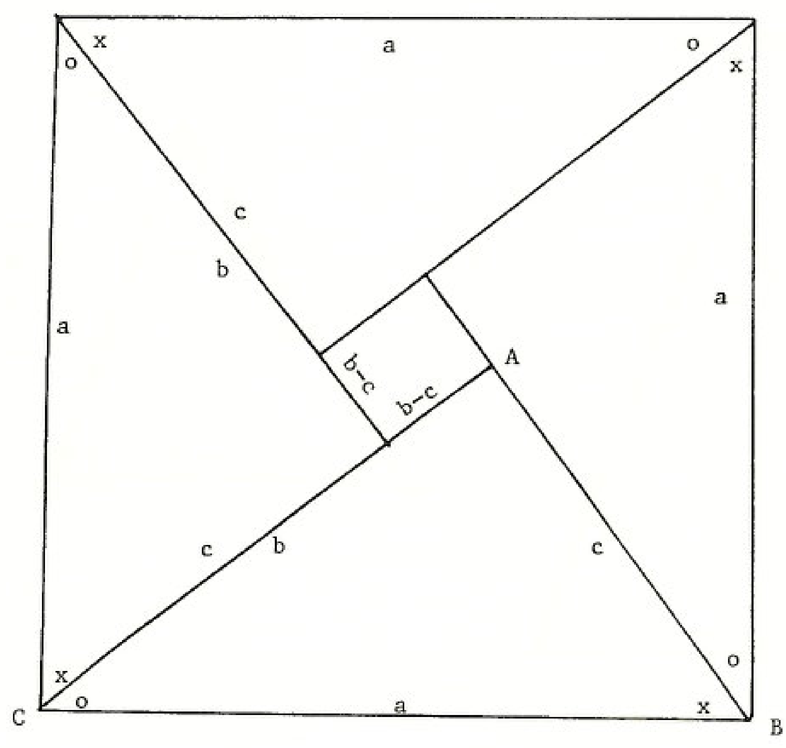


Figure 3

It is seen that the fitting together of the four triangles, in a symmetrical fashion, to form a perfect square is possible only because the angles o and x are complementary and so form a right-angle when adjacent. Also the four triangles almost fill the square, leaving only a square space at the centre, and it is easily seen that this central square has sides of length equal to $(b-c)$. We can now determine the area of the large square in two different ways, remembering that the area of each right-angled triangle is half the area of a rectangle whose sides are of length b and c. (This is another almost self-evident result far flat space, deducible from Euclid's assumption), and the large square is based on the hypotenuse, of length a, of each of the four triangles. Thus $a^2 = 4\left(\frac{1}{2}bc\right) + (b-c)^2 \ = b^2 + c^2$.

3. A demonstration using similar triangles We require here a property of similar triangles: that the ratio of the lengths of corresponding sides of such triangle is the same for any pair of such sides. Thus for the similar (or equiangular) triangles LMN and EFG, as in Figure 5, where

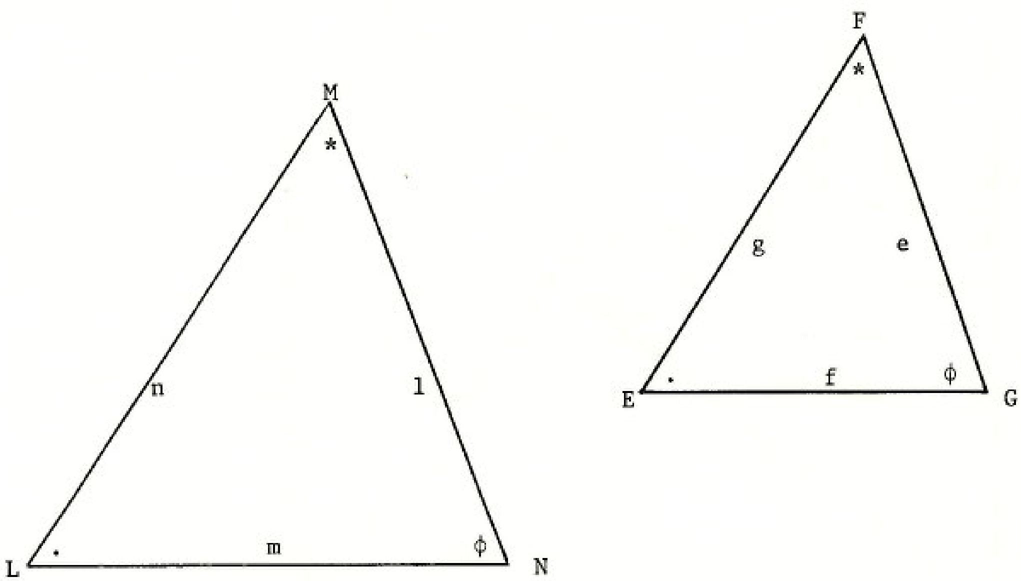
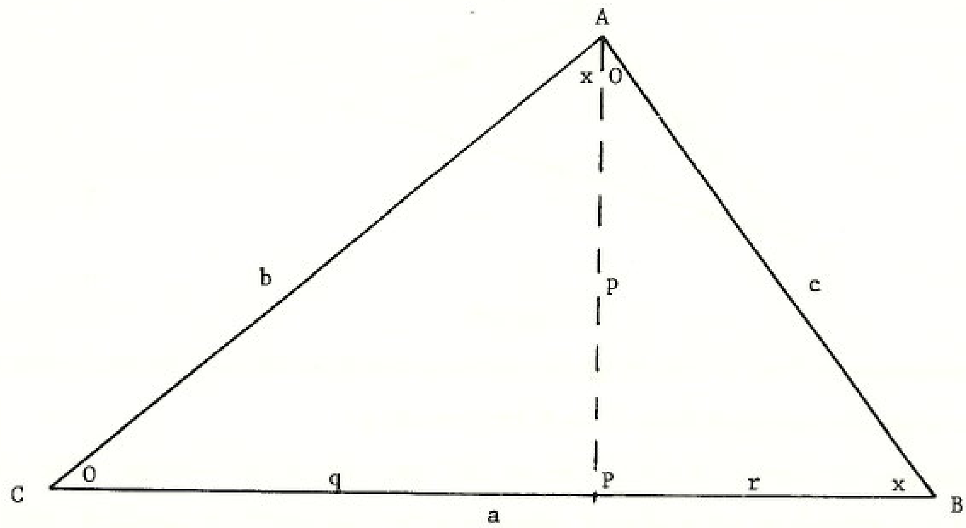


Figure 4

for example, LM and EF (opposite the equal angles at N and G respectively) are corresponding sides, so that $\dfrac{n}{g} = \dfrac{l}{e} = \dfrac{m}{f}$. Now, consider a right-angled triangle, ABC, as in Figure 5, with a line AP at right-angles to the hypotenuse. It is seen that $\triangle APC$ and

Figure 5



$\triangle ABC$ are also right-angled triangles and indeed similar to $\triangle ABC$. Hence, considering the corresponding sides of $\triangle APC$ and $\triangle BAC$, we have $\frac{q}{b} = \frac{b}{a}$, so that $aq = b^2$; and for $\triangle APB$ and $\triangle BCA$, $\frac{r}{c} = \frac{c}{a}$, so that $ar = c^2$. Hence $a(q+r) = b^2+c^2$, that is, $a^2 = b^2+c^2$. This is probably the neatest and most elegant demonstration of the Pythagoras theorem, and a 'proof' in the spirit of the Euclidian method. However, it depends on a property of similar triangles which is considered very late in the Euclidian system, and so is not available for underpinning the Pythagoras theorem which is considered much earlier. This may be because the latter theorem is vital for the development of circle geometry, etc.; whereas the proportionality property of similar triangles, though apparently self-evident, is difficult to deduce and, indeed, Euclid's proof of this property is highly suspect.

4. A demonstration using vector algebra

The use of vectors, as directed line-intervals or displacements, would appear to be particularly suited to demonstrate geometrical properties. However, their definitions imply and invoke much of Euclidean geometry — its assumptions and results — so their use is already heavily theory-laden. Nevertheless, a new approach to demonstrating a geometrical property may be more dramatic and convincing than a more formal proof.

For the purposes of the Pythagoras theorem, we need two properties of vectors:

(i) that the sum, $\underset{\sim}{a} + \underset{\sim}{b} = \underset{\sim}{c}$, illustrated by Figure 6, means that the combination

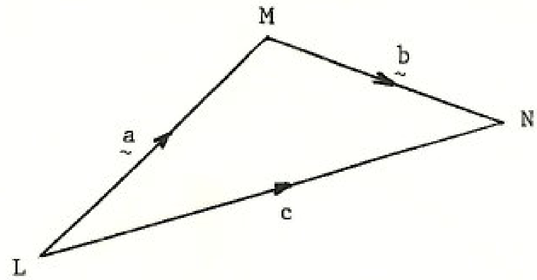


Figure 6

of two displacements, from L and M by the vector $\underset{\sim}{a}$ and from M to N by the vector $\underset{\sim}{b}$, is equal to a single displacement from L to N by a vector $\underset{\sim}{c}$.

We also require the notion of the scalar or 'dot' product of two vectors. The scalar product, $\underset{\sim}{a}.\underset{\sim}{b}$, is defined by $\underset{\sim}{a}.\underset{\sim}{b} = ab\cos\theta$, where a is the magnitude (or modulus) of $\underset{\sim}{a}$, b is the magnitude of $\underset{\sim}{b}$, and θ is the angle between the directed line-intervals corresponding to these vectors as displayed in Figure 7.

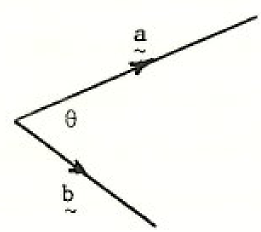


Figure 7

It follows that the scalar product is commutative — that is $\underset{\sim}{a}.\underset{\sim}{b} = \underset{\sim}{b}.\underset{\sim}{a}$, and that the distributive law, $\underset{\sim}{a}.(\underset{\sim}{b} + \underset{\sim}{c}) = \underset{\sim}{a}.\underset{\sim}{b} + \underset{\sim}{a}.\underset{\sim}{c}$, holds as it does for ordinary addition and multiplication involving numbers.

In particular, the definition of the scalar product means that $\underset{\sim}{a}.\underset{\sim}{a} = a^2$, and that $\underset{\sim}{a}.\underset{\sim}{b} = 0$ if the directed displacements corresponding to $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are at right-angles.

Now consider the right-angled triangle ABC, portrayed vectorially as in Figure 8.

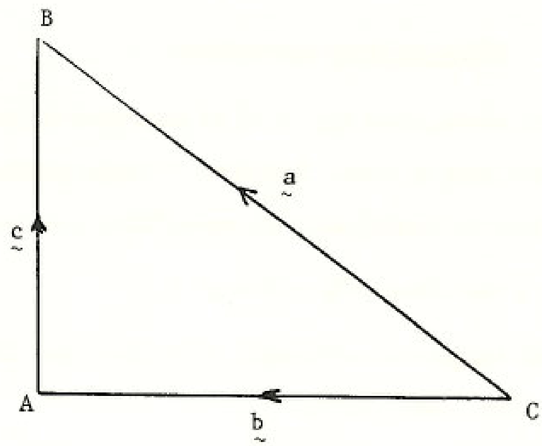


Figure 8

It is seen that:-

$$\underset{\sim}{a} = \underset{\sim}{b} + \underset{\sim}{c}$$

$$\text{so that} \quad \underset{\sim}{a}.\underset{\sim}{a} = (\underset{\sim}{b} + \underset{\sim}{c}).(\underset{\sim}{b} + \underset{\sim}{c})$$

$$= (\underset{\sim}{b} + \underset{\sim}{c}).b + (\underset{\sim}{b} + \underset{\sim}{c}).c$$

$$= b.(\underset{\sim}{b} + \underset{\sim}{c}) + \underset{\sim}{c}.(\underset{\sim}{b} + \underset{\sim}{c})$$

$$= \underset{\sim}{b}.\underset{\sim}{b} + \underset{\sim}{b}.\underset{\sim}{c} + \underset{\sim}{c}.\underset{\sim}{b} + \underset{\sim}{c}.\underset{\sim}{c},$$

$$\text{whence} \quad a^2 = b^2 + c^2$$

$$\text{since} \quad \underset{\sim}{b}.\underset{\sim}{c} = \underset{\sim}{c}.\underset{\sim}{b} = 0$$

for the case of our right-angled triangle. Essentially, the proof depends on the purely geometric notion that a straight line has zero projection on one at right-angles to it, and projects fully on one parallel to it.

Note also that it follows almost immediately that if the angle at A in Figure 8 is $\theta \neq 90°$, then we obtain an important result applying to all triangles in Euclidean space, viz

$$a^2 = b^2 + c^2 - 2bc\cos\theta$$

[How does the minus sign come about in the vectorial derivation of this result?]

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