SOLUTIONS OF PROBLEMS 805 - 816

Q.805 Solve for x and y:

$$\sqrt{x+y}+\sqrt{x-y}=5\sqrt{x^2-y^2}$$

$$\frac{2}{\sqrt{x+y}}-\frac{1}{\sqrt{x-y}}=1$$

ANSWER Dividing the first equation by $\sqrt{x^2-y^2}=\sqrt{x+y}\sqrt{x-y}$ gives $\frac{1}{\sqrt{x-y}}+\frac{1}{\sqrt{x+y}}=5.$

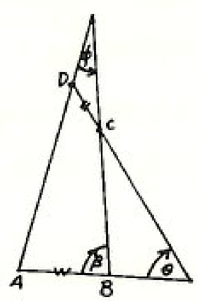
Adding this to the second equation:- $\frac{3}{\sqrt{x+y}}=6$

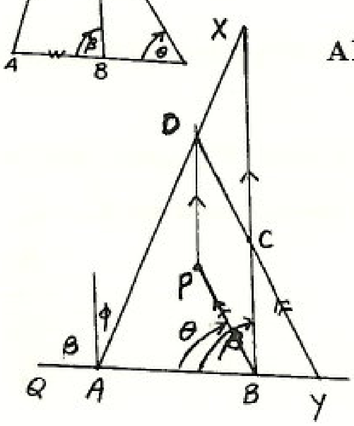
$$\therefore \frac{1}{\sqrt{x+y}}=2, \text{ and } \frac{1}{\sqrt{x-y}}=5-2=3.$$

These yield $x+y=\frac{1}{2^2}$; $x-y=\frac{1}{3^2}$; whence on adding $2x=\frac{13}{36}$, $x=\frac{13}{72}$, and on subtracting $2y=\frac{5}{36}$, $y=\frac{5}{72}$.

Thus the only solution is $(x,y)=\left(\frac{13}{72},\frac{5}{72}\right)$.

Correct Solution: James Choi (Trinity Grammar School)





Q.806 Show how to construct a quadrilateral $ABCD$, being given the lengths AB and CD, the angle $A\hat{B}C$, and the angles made by producing both pairs of opposite sides.

ANSWER Refer to figures 1 and 2.

On a straight line QY construct a segment AB of the given length. Using the given angles, β and ϕ, construct lines BX and AX having $\angle ABX=\beta$ and $\angle QAX=\beta+\phi$, intersecting at X.

Using the given angle θ construct P inside $\triangle ABX$ with $\angle PBA=\theta$ and BP equal to the given length of CD.

Through P construct a line parallel to BX intersecting AX at D, and then a line through D parallel to PB intersecting BX at C, and AB produced at Y.

Proof Since $BP\ DC$ is a parallellogram, $CD = BP$, the given length, and $\angle DYC = \angle PBA$ (corresponding angles) $= \theta$ (the given angle).
Also $\angle AXB = \angle QAX - \angle ABX = (\beta + \phi) - \beta = \phi$ (given).
Obviously AB has been constructed of the given length, and $\angle ABX$ equal to the given angle.
(For the construction to succeed the given quantities must satisfy the relations $\beta + \phi < 180°$; $\theta < \beta$; and $CD < \dfrac{AB\sin(\beta+\phi)}{\sin(\beta+\phi-\theta)}$. The first permits $\triangle ABX$ to be constructed with the correct angles at B and X. The other two permit P to be constructed inside this triangle.)

Correct Solution: J. Choi (Trinity Grammar School)

Q.807 A confirmed clock watcher drew lines on the face of a clock to mark the positions of the two hands. Some time later he noticed that the hands now trisected the angle between the marked lines. In how short a time could this have happened? How soon after 3.00 could one start such an experiment?

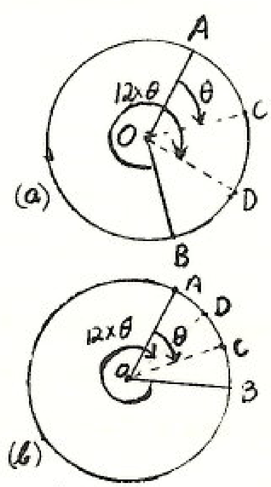


fig 1

ANSWER Let OA, OB represent the lines drawn on the clock-face, the hour hand having been at OA. After the hour hand has moved through $\theta°$, it reaches position OC (see fig.1 (a) or (b)). The minute hand having moved through $12 \times \theta°$ (since it moves 12 times as fast) has reached position OD. Since OC and OD are trisectors of $A\hat{O}B$, $D\hat{O}B = \theta°$.

Therefore $12 \times \theta° + \theta° = 360° \times n$ (where n is the number of complete revolutions that will have been made by the minute hand when it next reaches the position OB).

The smallest value of θ, corresponding to $n = 1$, is $\theta° = \dfrac{360°}{13}$. Since the hour hand moves 30° per hour it moves through $\frac{360°}{13}$ in $\frac{12}{13}$ hour. This is the shortest time that could have elapsed. $(= 55\frac{5}{13}$ minutes).

If fig 1(a) applies, the angle between the drawn lines is $3 \times \theta° = \frac{3 \times 360°}{13}$; if fig 1(b) applies, it is only $\frac{3}{2} \times \frac{360°}{13}$. At 3.00pm the minute hand is 90° behind the hour hand. To reach a point $\frac{3}{2} \times \frac{360°}{13}$ ahead of the hour hand, since it overtakes at the rate of 330° per hour, will require the elapse of $\frac{\frac{3}{2} \times \frac{360}{13} + 90}{330}$ hours, or of $\frac{(\frac{504}{13} + 90)}{330} \times 60$ minutes.

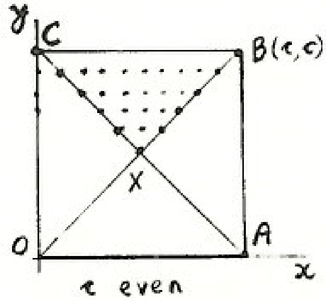
This is about 23.41 minutes, therefore the experiment could have started at 23.41 minutes past 3.00pm, but not sooner.

Q.808 Let $b_1, b_2, ..., b_n$ be any rearrangement of the positive numbers $a_1, a_2, ..., a_n$. Prove that $\frac{b_1}{a_1} + \frac{b_2}{a_2} + ... + \frac{b_n}{a_n} \geq n$.

ANSWER Let $x_k = \frac{b_k}{a_k}$ $k = 1, 2, \cdots, n$.

Since the arithmetic mean of the positive numbers $\{x_1, \cdots, x_n\}$ is not less than their geometric mean

$$\frac{x_1 + \cdots x_n}{n} \geq \sqrt[n]{x_1 x_2 \cdots x_n} = \sqrt[n]{\frac{b_1 \cdots b_n}{a_1 \cdots a_n}} = \sqrt[n]{1} = 1.$$



$$\therefore \frac{b_1}{a_1} + \cdots + \frac{b_n}{a_n} \geq n.$$

Q.809 How many different (i.e. non congruent) triangles can be constructed with side lengths in cms a, b, c given that a, b, c are all whole numbers, and $a < b < c \leq n$ where n is a given whole number.

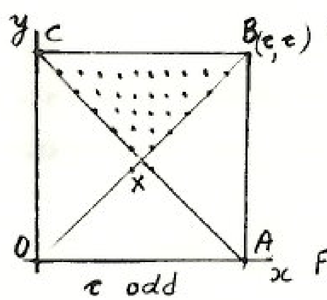


Fig 1

ANSWER Let $N(c)$ denote the number of different triangles with longest side c cms and other sides a, b cms where $a < b < c$, and a, b and c are all whole numbers.

$N(c)$ is the number of points with integer co-ordinates (a, b) inside the triangle BCX in the diagrams in Fig.1.

To see this, observe that since $a < b$ the points must lie above the diagonal $AC(x = y)$. Since $b < c$ they must lie underneath the line $BC(y = c)$.

Since to form a triangle we must have $a + b > c$ they must lie above the diagonal AC ($x + y = c$).

Since there are equal numbers of integer points in each of the 4 triangles in the diagrams, and there are $(c-1)^2$ integer points inside the square $OABC$ we have $N(c) = \frac{1}{4}[(c-1)^2 - D(c)]$ where $D(c)$ denotes the number of integer points inside the square on the two diagonals OB and AC. There are $(c-1)$ such points on each diagonal, so $D(c) = 2(c-1)$ when c is odd. However, when c is even $(\frac{c}{2}, \frac{c}{2})$ lies on both diagonals, and $D(c) = 2(c-1) - 1$.

Hence $N(c) = \frac{1}{4}[(c-1)^2 - 2(c-1) + 1] = \frac{1}{4}(c-2)^2$ when c is even, and

$N(c) = \frac{1}{4}[(c-1)^2 - 2(c-1)] = \frac{1}{4}[(c-2)^2 - 1]$ when c is odd.

Now let $F(n)$ denote the number of different scalene triangle with integer sides of which the longest does not exceed n.

$$\begin{aligned} F(n) &= N(1) + N(2) + N(3) + N(4) + \cdots + N(n) \\ &= \frac{1}{4}\sum_{c=1}^{n}(c-2)^2 - k \times \frac{1}{4} \end{aligned}$$

where k is the number of odd numbers in $\{1, 2, \cdots, n\}$. So $k = \frac{n}{2}$ if n is even, and $\frac{n+1}{2}$ if n is odd.

$$\therefore F(n) = \frac{1}{4} \times 1 + \frac{1}{4}\sum_{m=0}^{n-2} m^2 \quad - \begin{cases} \frac{n}{8} & (n \text{ even}) \\ \frac{n+1}{8} & (n \text{ odd} \end{cases}.$$

Using the identity $\sum_{m=0}^{N} m^2 = \frac{1}{6}N(N+1)(2N+1)$ (which is not too hard to prove by mathematical induction) we obtain

$$F(n) = \frac{1}{4} + \frac{1}{4}(n-2)(n-1)(2n-3) - \frac{n}{8} - \begin{cases} 0 & (n \text{ even}) \\ \frac{1}{8} & (n \text{ odd}) \end{cases}$$

$$F(n) = \begin{cases} \dfrac{2n^3 - 9n^2 + 10n}{24} = \dfrac{n(n-2)(2n-5)}{24} & (n \text{ even}) \\ \dfrac{2n^3 - 9n^2 + 10n - 3}{24} = \dfrac{(n-1)(2n-1)(n-3)}{24} & (n \text{ odd}). \end{cases}$$

Q.810 Solve for x the equation

$$\sin(18+\frac{3x}{2})^0 = 2\sin(54-\frac{x}{2})^0$$

ANSWER Since $\sin(180-\theta)^\circ = \sin\theta^\circ$, we can reformulate the question to read

$$\sin(180-(18+\frac{3x}{2}))^\circ = 2\sin(54-\frac{x}{2})^\circ$$

$$\sin 3y^\circ = 2\sin y^\circ \text{ where } y = 54-\frac{x}{2}$$

Since $\sin 3\theta = \sin\theta\cos 2\theta + \cos\theta\sin 2\theta = \sin\theta(1-2\sin^2\theta) + 2\sin\theta\cos^2\theta$

$$= 3\sin\theta - 4\sin^3\theta$$

$$\sin 3y = 2\sin y \Leftrightarrow 3\sin y - 4\sin^3 y = 2\sin y$$

$$\Leftrightarrow \sin y(4\sin^2 y - 1) = 0$$

$$\Leftrightarrow \sin y = 0 \text{ or } \sin y = \pm\frac{1}{2}$$

$$\Leftrightarrow y = 180\times n \text{ or } y = \pm 30 + 180n \ (n \text{ any integer})$$

$$\Leftrightarrow x = 108 \pm 360n, \text{ or } x = 48 \pm 360n, \text{ or } x = 168 \pm 360n \text{ where } n = 0,1,2,\cdots$$

Correct Solution: J. Choi (Trinity Grammar School)

Q.811 All digits of a whole number x are different, and they are in increasing order. Let $S(x)$ denote the sum of all the whole numbers which can be obtained by rearranging the digits of x (including x itself). Thus, if $x = 378$, $S(x) = 378 + 387 + 738 + 783 + 837 + 873 = 3996$. Find all numbers x such that $S(x) = 86658$.

ANSWER Let the digits of x be $d_1 d_2 \cdots d_k$, so $x = d_1 \times 10^{k-1} + d_2 \times 10^{k-2} + \cdots + d_k$. Of the $k!$ different arrangements of these digits, each d_j occurs $(k-1)!$ times in the 1st place, $(k-1)!$ times in the 2nd place, $\cdots$, and $(k-1)!$ times in the last place. Therefore when the $k!$ arrangements are added up, the digits in each column add to $(d_1 + d_2 + \cdots + d_k) \times (k-1)!$

$$\therefore S(x) = (d_1 + d_2 + \cdots + d_k) \times (k-1)! \times (10^{k-1} + 10^{k-2} + \cdots + 10^1 + 1).$$

If $k \geq 5$, $S(x) \geq (1+2+3+4+5) \times 4! \times 1\,1\,1\,1\,1 = 3999960$.

If $k \leq 3$, $S(x) \leq (7+8+9) \times 2! \times 1\,1\,1 = 5328$.

Hence if $S(x) = 86658$, x must have 4 digits and

$86658 = (d_1 + d_2 + d_3 + d_4) \times 3! \times 1111$;

whence $d_1 + d_2 + d_3 + d_4 = 13$.

By trial there are only three possible values of x:

viz. 1237, or 1246, or 1345.

Q.812 $P(x)$ is a polynomial with integer coefficients, and n, m are positive whole numbers.

Given that (i) $19 > n > m$

(ii) $P(m) = 0$, $P(n) = 39$, $P(19) = 187$

find m and n.

ANSWER Let $P(x) = a_0 + a_1x + \cdots + a_tx^t$ where $a_0, a_1, \cdots, a_t$ are integers.

Since for each positive integer k

$$c^k - d^k = (c-d)(c^{k-1} + c^{k-2}d + c^{k-3}d^2 + \cdots + d^{k-1}),$$

it follows that if c, d are any integers, then $c - d$ is a factor of

$P(c) - P(d) = a_1(c-d) + a_2(c^2 - d^2) + \cdots + a_t(c^t - d^t)$. Taking $(c,d) = (19, m)$, then $(19, n)$, then (n, m) we have

$19 - m$ is a factor of $P(19) - P(m) = 187 = 11 \times 17 \cdots (1)$

$19 - n$ is a factor of $P(19) - P(n) = 187 - 39 = 148 \cdots (2)$

$n - m$ is a factor of $P(n) - P(m) = 39 = 3 \times 13 \cdots (3)$

From (1), since $0 < m < n < 19$, $19 - m = 11$, or 17; and from (3), $n - m = 1$, or 3, or 13.

Since $19 - n = (19 - m) - (n - m)$, from (2) the only possibility is $19 - m = 17$, and $n - m = 13$ (because none of 11-1, 11-3, 17-1, or 17-3 is a factor of 148).

Hence the only solution is $m = 2, n = 15$.

[It is not too difficult to exhibit a polynomial $P(x)$ having all the stated properties.

$P(x) = (x-2)(2x-27) = 2x^2 - 31x + 54$ is one possibility].

Q.814 The "increasing" list of numbers $a_1, a_2, a_3, ..., a_k, ...$ is defined by

$a_1 = 1$, and $a_k^2 + a_{k+1}^2 = 2(\frac{a_k + a_{k+1}}{2} + 1)^2$ for $k \geq 1$.

(i) Prove that all terms in the list are whole numbers.

(ii) Find a_{1990}.

ANSWER The word "increasing" was inadvertently omitted in the statement of this problem. We first solve the amended question.

The given relationship between a_k and a_{k+1} is equivalent to
$a_{k+1}^2 - 2(a_k+2)a_{k+1} + (a_k^2 - 4a_k - 4) = 0$.
Solving this quadratic in a_{k+1} gives

$$a_{k+1} = a_k + 2 \pm \sqrt{8(a_k+1)} \qquad (1)$$

For an increasing sequence, $a_k \geq a_1 = 1$ and we must take
$a_{k+1} = a_k + 2 + \sqrt{8(a_k+1)}$ to ensure $a_{k+1} > a_k$.
Calculating the first few terms of the list we obtain

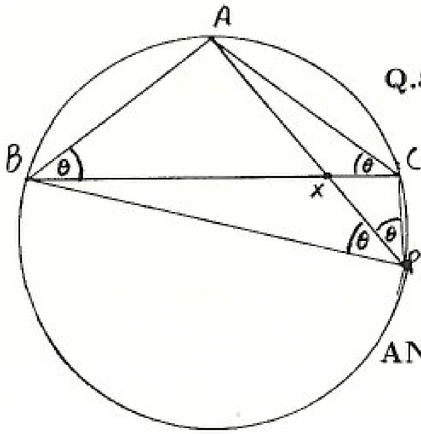
n	1	2	3	4	5	6
a_n	1	7	17	31	49	71

Note that for these small values of $n, a_n = 2n^2 - 1$.
[Perhaps you might notice first that the differences of successive numbers a_n form an arithmetic progression. That observation, $a_{k+1} - a_k = 4k + 2$, can be used with (1) to obtain the formula we have just exhibited.]

It is not too difficult to check by mathematical induction that this formula continues to apply for all positive integers n. Thus, from (1), if $a_k = 2k^2 - 1$

$$a_{k+1} = 2k^2 + 1 + \sqrt{16k^2} = 2(k+1)^2 - 1.$$

Thus all terms in the list are whole numbers, and $a_{1990} = 2 \times 1990^2 - 1$.
[Without the word "increasing" in the question, (1) gives two values of a_{k+1} in terms of a_k, so the list is not determined uniquely. Thus $a_2 = -1$, or 7. Taking $a_2 = -1, a_3 = +1 \pm 0$. Taking $a_2 = 7, a_3 = 1$ or 17. In fact, (i) is still true, the possible values of a_k being $\{2n^2 - 1 : n = 0, 1, 2, \cdots\}$. If $a_k = 2n^2 - 1$ then a_{k+1} is either $2(n+1)^2 - 1$ or $2(n-1)^2 - 1$. The possible values of a_{1990} would then be $\{2n^2 - 1 : \text{where } n = 0, 2, 4, \cdots, 1990\}$.]



Q.815 The isosceles triangle ABC is obtuse angled at A and P is a point on the circumcircle such that
$PB + PC = 2AB$.
Prove that $2AX = BC$ (where X is the point of intersection of AP and BC).

ANSWER Note that $A\hat{P}C = A\hat{B}C = A\hat{C}B = A\hat{P}B$.

$\therefore \triangle AXC \sim \triangle ACP$ since they are equiangular.

$$\therefore PC = AC \times \frac{CX}{AX} \qquad (1)$$

Similarly from $\triangle AXB \sim \triangle ABP$ we find $PB = \frac{AB \times BX}{AX}$ (2)

Adding (1) and (2), after replacing AC by AB in (1)

$$2AB = PC + PB = AB\frac{(BX + CX)}{AX}$$

$$\therefore 2AX = BC$$

Correct Solution: J. Choi (Trinity Grammar School)

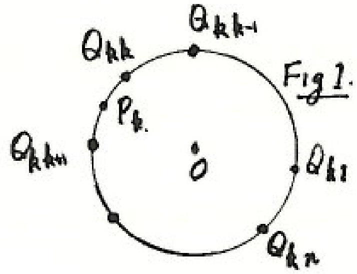
Q.816 Let $P_1, P_2, ..., P_n$ be any n distinct points in the plane. Show that it is always possible to dissect the plane into n regions $R_1, R_2, ..., R_n$ all congruent to one another, such that P_i lies inside R_i for $i = 1, 2, ..., n$.

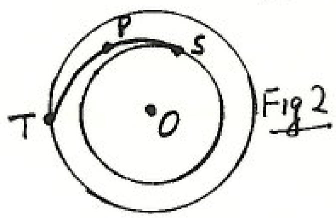
ANSWER Choose a point O in the plane which is not equidistant from any two of the points $P_k(k = 1, 2, \cdots n)$ (i.e. O is any point not lying on the perpendicular bisector of P_iP_j for any i, j). Let us relabel the points so that the lengths $OP_1, OP_2, \cdots OP_n$ are in increasing order. Draw a circle C_k, centre O passing through P_k for $k = 1, 2, \cdots n$.

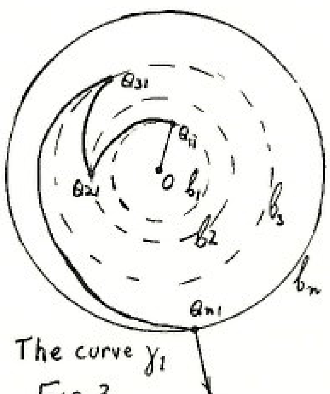
For each k, divide the circle C_k into n equal arcs by points $Q_{k_1}, Q_{k_2}, \cdots, Q_{k_n}$ in anticlockwise order around C_k, with P_k lying in the arc $Q_{kk}Q_{kk+1}$ for $k = 1, 2, \cdots,$

$n - 1$ and P_n lying in the arc $Q_{nn}Q_{n1}$.

We shall call a curve from S to T "good" if the distance from O to P increases steadily as P describes the curve. (See fig.2.) Obviously it lies in the annulus between circles centre O passing through S and T.







The curve γ_1

Fig 3

Now join O to Q_{11} by a straight line segment, join Q_{k1} to $Q_{(k+1)1}$, by a good arc for $k = 1, 2, \cdots, n-1$ and draw a ray from Q_{n1}. Let γ_1 be the connected good curve so formed. (see fig.3).

Now let γ_k, $k = 2, 3, \cdots n$ be the curve obtained by rotating γ_1 anticlockwise about O through $\frac{k-1}{n}$ of a revolution. Clearly γ_k contains the points $O, Q_{k1}, Q_{k2}, \cdots, Q_{kk}, \cdots, Q_{kn}$.

Denote by $\mathcal{R}_k$ the region in the plane between γ_k and γ_{k+1} for $k = 1, 2, \cdots,$
$n-1$ and by $\mathcal{R}_n$ the region between γ_n and γ_1. Since a rotation of the plane about O through $\frac{1}{n}th$ of a revolution takes $\mathcal{R}_k$ onto $\mathcal{R}_{k+1}$ (or $\mathcal{R}_n$ onto $\mathcal{R}_1$) all these regions are congruent. Also by construction the arc $Q_{kk}Q_{kk+1}$ of C_k lies in $\mathcal{R}_k$, and hence P_k lies in $\mathcal{R}_k$ for $k = 1, 2, \cdots, n$.