GRAPHS, BULBS AND MATRICES

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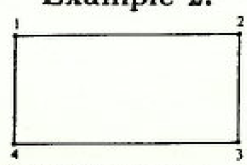
This article is a revised version of a talk given to the Mathematics Club at the Technion, Israel's Technological University and subsequently printed in Etgar-Gilianot Mathematica, the Israeli version of Parabola. One of the problems discussed in the article was a question concerning the switching of electric bulbs situated at the vertices of a graph, and the solution provided a nice illustration of the use of matrix methods and the theory of equations in such a "real-life" problem.

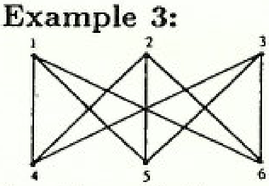
By a **graph** we mean a network consisting of a finite number of vertices where some of the vertices are connected by an edge. Two vertices connected by an edge are said to be **neighbours**.

Example 1:

Example 2:



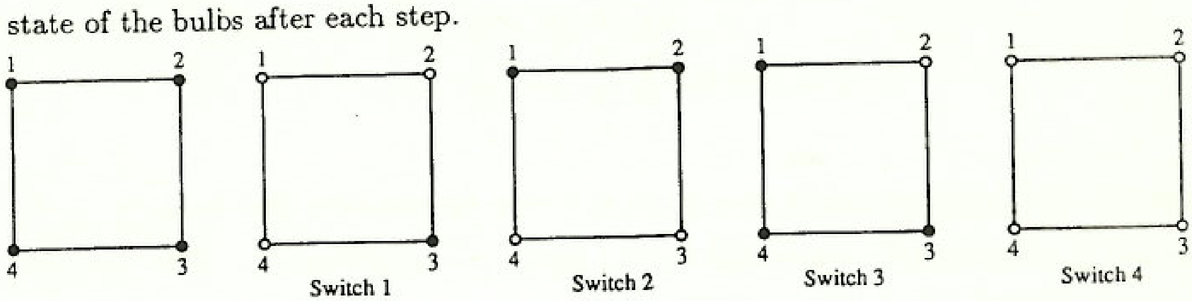
Example 3:



The first example is of a graph which consists of two connected vertices. In the second example we have a graph with four vertices. Here vertex No.1 is not a neighbour of vertex No.3 and vertex No.2 is not a neighbour of vertex No.4. All other pairs of vertices are connected. The third example has six vertices and each of the first three vertices is a neighbour of each of the last three vertices.

Our problem is this: We are given an arbitrary graph G with n vertices. At each of the n vertices there is a bulb and an on/off switch. Turning a switch activates the bulb at it and its neighbours, that is if the bulb was off it will be lit and if it was on, it will be off. Initially all bulbs are off. Prove that operating some of the switches all bulbs can be turned on simultaneously.

In the first two examples this can be achieved fairly simply. The first one is trivial: operating switch 1, both bulbs are lit up which is exactly what we want to achieve. The same is true of a graph of any size in which every pair of vertices is connected; operating any one of the switches will light up all the bulbs simultaneously. In the second example we must operate all the switches 1,2,3,4 in succession. The following diagram shows the state of the bulbs after each step.



We end up with all bulbs lit up. The solution for the third example will be given later.

In the case of an arbitrary graph it is by no means obvious that the desired end can be achieved. Let us formulate the problem in a more arithmetic form. Suppose we have a graph G with n vertices and we operate k switches at the vertices $s_1, \cdots, s_k$. The desired end result "all bulbs lit" will be achieved if it is true that for each vertex v the number of those neighbours of v, including v itself, which belong to the set $S = \{s_1, \cdots, s_k\}$ is odd. For then the number of times the bulb at v is turned on or off is odd hence if it started "off" it will end up "on". Is it possible for any graph G to find such a subset of switches? Clearly all that matters is the subset S of switches to be operated, but not the order in which we operate them.

To complete the "arithmetization" we formulate the problem in matrix language. We associate with G an $n \times n$ matrix $A = (a_{ij})$, that is an array of n^2 numbers arranged in a tableau of n rows and n columns, with a_{ij}, the entry in the i-th row and j-th column, as follows:

$$a_{ii} = 1 \quad i = 1, 2, \cdots, n$$

$$a_{ij} = 1 \quad \text{if } i \text{ and } j \text{ are neighbours}$$

$$a_{ij} = 0 \quad \text{otherwise.}$$

A is usually called the **incidence matrix** of the graph G. For instance the incidence matrix of the graph of Example 1 is

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

and the incidence matrix of the graph of Example 2 is

$$A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$$

Note that the incidence matrix of a graph is always **symmetric**, that is $a_{ij} = a_{ji}$ for all pairs (i,j) because if i is a neighbour of j then j is a neighbour of i. The "principal diagonal" of entries a_{ii} consists of all 1's.

Now select a subset $\{s_1, \cdots, s_k\}$ of the vertex set $\{1, 2, \cdots, n\}$, representing those switches which we want to operate. Define the numbers $x_1, x_2, \cdots, x_n$ as follows: set

$$x_j = 1 \quad \text{if } j \text{ equals any one of the } s_i$$

$$x_j = 0 \quad \text{otherwise.}$$

For each i form the sum

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = e_i$$

Here $a_{ij} = 0$ if the vertex j is not a neighbour of i and $x_j = 0$ if j is not one of the selected switches. Therefore $a_{ij}x_j = 1$ exactly when j is either a neighbour of i or equals i, and j is also one of the selected switches s. Our aim is then to determine the collection $(x_1, x_2, \cdots, x_n)$ of 0's and 1's such that each of the numbers e_i turns out to be odd. So we are finished if we can prove the following theorem:

Theorem. *Let $A = (a_{ij})$ be the incidence matrix of a graph G with n vertices, that is a symmetric matrix of 0's and 1's, with $a_{ii} = 1$ and $a_{ji} = a_{ij}$ for all pairs (i,j). Then there is a collection $(x_1, x_2, \cdots, x_n)$ of 0's and 1's such that*

$$e_i = a_{i1}x_1 + \cdots + a_{in}x_n \quad \text{is odd for every } i = 1, 2, \cdots, n$$

Notice that the e_i are positive integers (a sum of 1's) or perhaps 0, but their actual value is of no interest, only their "parity", that is their being even or odd. This leads us to the next important step: we regard 0, 1 and e_i not as ordinary numbers but as numbers modulo 2, that is members of an arithmetic that contains only two elements, namely 0 and 1 and which has the following addition and multiplication tables:

+	0	1
0	0	1
1	1	0

×	0	1
0	0	0
1	0	1

The only unusual rule is $1 + 1 = 0$, expressing the fact that the sum of two odd numbers is even. For instance the sum of an even number of 1's is 0, the sum of an odd number of 1's is 1. The set of these new numbers is usually denoted Z_2. They form a **field**, meaning by this that they share with ordinary real numbers the usual laws of arithmetic. The readers surely know that every three real numbers x, y, z satisfy the following:

a. Addition is commutative

$$x + y = y + x$$

b. Addition is associative

$$x + (y + z) = (x + y) + z$$

c. There exists a unique number, 0, satisfying

$$x + 0 = x$$

d. There exists a unique number, $-x$, such that

$$x + (-x) = 0$$

e. Multiplication is commutative

$$xy = yx$$

f. Multiplication is associative

$$x(yz) = (xy)z$$

g. There exists a unique number, 1, which is different from 0 and satisfies

$$x1 = x$$

h. For every $x \neq 0$ there exists a unique number x^{-1} such that $xx^{-1} = 1$

i. Addition and multiplication are related through the distributive law

$$x(y+z) = xy + xz.$$

The above nine properties are the axioms in the definition of a field. To be more precise: A field is a set F with two operations called addition and multiplication. The addition operation associates with every two elements $x, y \in F$ an element $x + y \in F$ and the multiplication associates with x, y an element $xy \in F$, in such a way that the above mentioned nine properties are satisfied.

The set of real numbers is an example of a field and so is the set of rational numbers and complex numbers, on the other hand the set of integers is not a field (why?). More importantly for our present purposes also Z_2 is a field. Readers are invited to check that Z_2 with addition and multiplication as defined above indeed satisfies the nine axioms for a field. Why is it important to know that Z_2 is a field? Let us first formulate the theorem to be proved in terms of Z_2. The theorem states that if $A = (a_{ij})$ is a symmetric $n \times n$ matrix "over Z_2", that is $a_{ji} = a_{ij} \in Z_2$, $a_{ii} = 1 \in Z_2$ for all i, j, then the n equations

$$a_{i1}x_1 + \cdots + a_{in}x_n = 1, \; i = 1, 2, \cdots, n$$

have a solution $(x_1, \cdots x_n)$ with $x_i \in Z_2$.

What have we gained with this new formulation of the theorem? First of all the odd numbers e_i in the original formulation have been replaced simply by 1. More importantly, we have converted the problem into one of the basic problems of algebra namely solution of a system of n equations in n unknowns, with coefficients a_{ij} in a field. It is here that the importance of Z_2 being a field enters the argument. When has such a system of equations a solution $(x_1, \cdots, x_n)$? Algebra gives a simple answer: the system has a solution if and only if it is true that any "relation" that the left hand expressions satisfy, is also satisfied by the "numbers" (elements of the field) on the right hand side. Actually algebra tells us much more: it gives us all the solutions if the solvability condition is satisfied, and the number of equations need not even be the same as the number of unknowns. Those refinements need not concern us here.

What do we mean by "relations"? Instead of giving an accurate definition, we shall be satisfied with showing an example which will illuminate sufficiently the meaning of the condition. Consider the following three equations in three unknowns (with coefficients ordinary numbers): $2x_1 + x_2 = 1$, $x_2 + 2x_3 = 1$, $x_1 + x_2 + x_3 = 2$. This system has no possible solutions; for if we add the first two equations and subtract from them twice the third equation we get 0 on the left hand side (irrespective of what x_1, x_2, x_3 are) but $1 + 1 - 2 \times 2 = -2$ on the right hand side, a contradiction. Had we taken 1 instead of 2 on the right hand side of the last equation,, no contradiction would have arisen, and indeed $x_1 = x_2 = x_3 = \frac{1}{3}$ is then a solution (there are many others, for instance $x_1 = 1, x_2 = -1, x_3 = 1$ is also a solution).

The solvability condition can now be applied to our Theorem. What we need to show is that every combination of the n equations in the theorem which gives 0 will also give 0 on the right hand side. First we note that since the only numbers in Z_2 are 0 or 1, a "combination" merely means that we select a subset of the equations and add them up. The combination results in 0 if for each x_j the number of those coefficients a_{ij} which are equal to 1 when i is one of the selected equations, is even. For the sum of an even number of 1's in Z_2 is 0. To simplify notation let us assume that the sum of the 1st, 2nd, $\cdots k$-th equations (the left hand sides) is 0. This is no restriction in generality; we can achieve it by simply renumbering the vertices of our graph G. Consider the smaller submatrix (a_{ij}) where i, j go from 1 to k. We know that in each column the number of 1's is even, and so the total number of 1's is even. But we also know that each diagonal element a_{ii} is 1 and the number of 1's under the diagonal is the same as above the diagonal (since the matrix is symmetric) hence the total number of off-diagonal terms $a_{ij}, i \neq j$ which are equal to 1 is even. It follows that the number of diagonal terms, that is k, must be even if the selected combination is to give 0, hence the sum of the k 1's on the right hand side of the equation is also 0, as required by the solvability condition. The final conclusion is that the system of equations (over Z_2) in the theorem does have a solution and this was all that was required for the switching problem to have a solution.

You may be excused for thinking that perhaps the algebraic machinery used in the proof is too "heavy" for such a simple switching problem. If so, you are invited to submit a simpler solution; both Etgar-Gilionot and Parabola will be happy to publish alternative solutions submitted by readers.

The algebraic method certainly has one great merit: it provides a systematic way to find a solution. Let us demonstrate how it works for our example 3. The adjacency matrix of this graph is

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}$$

and the system of equations to be solved (in $\mathbf{Z}_2$) is

$$x_1 + x_4 + x_5 + x_6 = 1$$

$$x_2 + x_4 + x_5 + x_6 = 1$$

$$x_3 + x_4 + x_5 + x_6 = 1$$

$$x_1 + x_2 + x_3 + x_4 = 1$$

$$x_1 + x_2 + x_3 + x_5 = 1$$

$$x_1 + x_2 + x_3 + x_6 = 1$$

From the first three equations we find $x_1 = x_2 = x_3$ and from the last three $x_4 = x_5 = x_6$. Replacing x_2 and x_3 by x_1, x_5 and x_6 by x_4, and observing that $x_1 + x_1 + x_1 = x_1$, $x_4 + x_4 + x_4 = x_4$ in Z_2, all six equations reduce to $x_1 + x_4 = 1$. One solution is $x_1 = 1$, $x_4 = 0$ and we get $(x_1, x_2, x_3, x_4, x_5, x_6) = (1, 1, 1, 0, 0, 0)$. That is, we operate switches 1,2 and 3. You may easily convince yourself that this indeed does what we want. An alternative solution is $x_1 = 0$, $x_4 = 1$ (there are no others), leading to the operation of switches 4, 5 and 6.

Finally to test your skill, here is a graph on 10 vertices (well known to graph theorists under the name of Petersen's graph). Which switches to operate? Answers are welcome.

Example 4