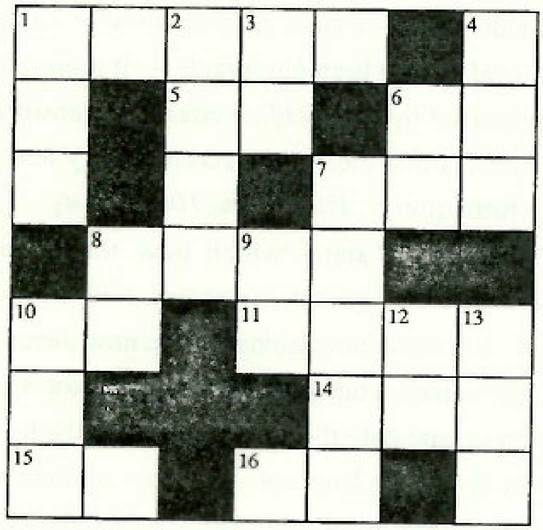
PUZZLER OF THE MONTH

Here is a puzzle to end all puzzles. It is said that only a genius can complete it successfully. There is no guess work involved. The answers are obtained through sound reasoning and from the key information supplied.

The puzzle was invented when there were 20 shillings to the English pound. 1 rood=$\frac{1}{4}$ acre, 1 acre=4840 sq. yds, 1 mile=1760 yds. One of the cross numbers is the same as one of the down numbers. One number in the puzzle (relating to something quite different) happens to be the area in roods of the rectangular field known as Dogsmead.

What year was the Puzzle invented?



ACROSS

1. Area of Dogsmead in square yards.
5. Age of farmer Dunk's daughter Martha.
6. Difference between length and breadth (in yards) of Dogsmead.
7. Number of roods in Dogsmead times. 9 DOWN.
8. Year when Little Piggley came into occupation of the Dunks.
10. Farmer Dunk's age.
11. Year of birth of Mary, Dunk's youngest.
14. Perimeter (yards) of Dogsmead.
15. Cube of Dunk's walking speed (M.P.H.).
16. 15 ACROSS minus 9 DOWN.

DOWN

1. Value of Dogsmead in shillings per acre.
2. Square of Mrs. Grooby's age.
3. Mary's age.
4. Value of Dogsmead in English pounds.
6. Age of Dunk's firstborn, Ed, who will be twice as old as Mary next year.
7. Square of the number of yards which is the breadth of Dogsmead.
8. Number of minutes Farmer Dunk takes to walk 1 & 1/3 times around Dogsmeac
9. See 10 DOWN.
10. 10 ACROSS times 9 DOWN.
12. Sum of digits in 2nd column plus one.
13. Length of tenure (in years) of little Piggley by the Dunks.

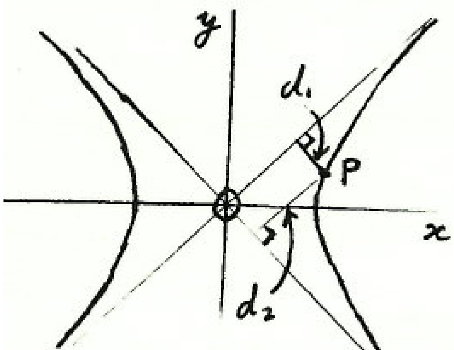
PROBLEM SECTION

You are invited to submit solutions to any or all of the following problems, accompanied by your name, school and year or form. Solutions of these problems will appear in the next issue of Parabola; your solution(s) may be used if they are received in time.

Q.829 (i) Let c be any integer. Show that the remainder when c^2 is divided by 4 cannot be either 2 or 3.

(ii) Let x be a positive integer, $A = 2x - 1$, $B = 5x - 1$, $C = 13x - 1$. Show that any two of A, B, C may be perfect squares, but that it is impossible that all three are squares.

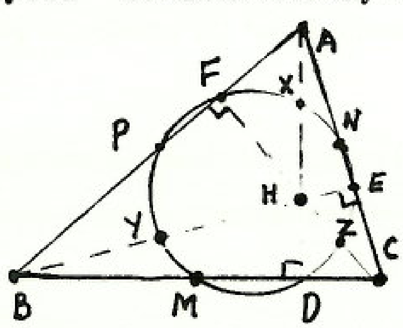
Q.830 The curve with the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is called a hyperbola.



The straight lines $\frac{x}{a} \pm \frac{y}{b} = 0$, through the origin are called asymptotes. If $a = b$ the asymptotes are at righ-tangles, and the hyperbola is called a rectangular hyperbola. Let the perpendicular distances from a point P on a rectangular hyperbola to the asymptotes be d_1, and d_2. Show that $d_1 \times d_2 = 2a^2$.

Deduce that by taking new axes along the asymptotes and adjusting the unit of length appropriately, any rectangular hyperbola can be given the Cartesian equation $XY = 1$.

Q.831 Let $\triangle ABC$ be any triangle.



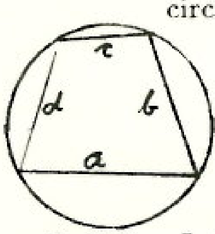
Let MN, P be the mid points of the sides, D, E, F the feet of the altitudes, H the orthocentre and X, Y, Z the mid points of HA, HB, HC respectively. Show that the points $M, N, P, D, E, F, X, Y, Z$ all lie on one circle. This is called the nine point circle of $\triangle ABC$.

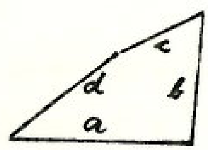
Q.832 Let A, B, C be distinct points all lying on a rectangular hyperbola. Show that the centre of the hyperbola (the point of intersection of the asymptotes) lies on the nine point circle of $\triangle ABC$.

Q.833 If x is any positive integer, $f(x)$ denotes the new integer obtained when the last digit of x (using the usual decimal representation) is transferred to the other end; e.g. $f(1356) = 6135$. Find the smallest integer such that $f(x) = 7 \times x$.

Q.834 Let $N(n)$ denote the number of different solutions in non- negative integers w, x, y, z of the equation $w^2 + x^2 + y^2 + z^2 = 3 \times 2^n$.
For example, $N(0) = 4$ since the only solutions of $w^2 + x^2 + y^2 + z^2 = 3$ are $(w, x, y, z) = (1,1,1,0)$ or $(1,1,0,1)$ or $(1,0,1,1)$ or $(0,1,1,1)$. Find $N(1991)$.

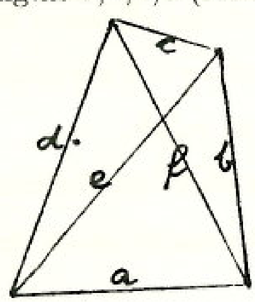
Q.835 It is known that the region of maximum area having given perimeter p is the circular disc with radius $\frac{p}{2\pi}$.





Assuming this or otherwise prove that of all quadrilaterals with sides of given lengths a, b, c, d that of maximum area is cyclic.

Q.836 Let e, f be the lengths of the diagonals of a cyclic quadrilateral with sides of lengths a, b, c, d (see figure).



Show that (i) $e(ab + cd) = f(ad + bc)$
(Hint: See Q.818).

(ii) $e^2 = \dfrac{(ac+bd)(ad+bc)}{(ab+cd)}$

$f^2 = \dfrac{(ac+bd)(ab+cd)}{(ad+bc)}$

(You may assume Ptolemy's Theorem: $ef = ac + bd$).

Q.837 Let the sides in order round any quadrilateral have lengths a, b, c, d, where $a^2 + c^2 = b^2 + d^2$. Prove that the area of the quadrilateral is half the product of the lengths of the diagonals.

Q.838 Rods of lengths 60, 52, 39, and 25 units are joined together at their end points in any order to make a plane quadrilateral: Calculate the maximum possible value of its area. (Note that $60^2 + 25^2 = 50^2 + 39^2$).

Q.839 Prove that $\sqrt[3]{40 + \sqrt{1573}} + \sqrt[3]{40 - \sqrt{1573}}$ is **exactly** equal to 5.