SOLUTIONS OF PROBLEMS 817 - 828

Q.817 Find all integers x, y such that $x(3y-5) = y^2+1$.

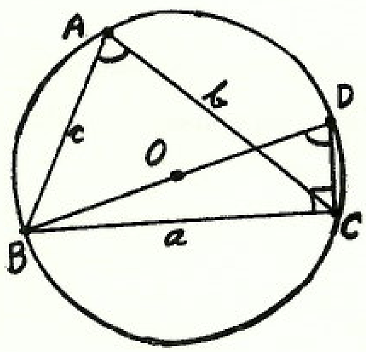
ANSWER $x = \dfrac{y^2+1}{3y-5} = \dfrac{1}{3}y + \dfrac{5}{9} + \dfrac{\frac{34}{9}}{3y-5}$ after a routine division of polynominals.

$\therefore 9x = 3y + 5 + \dfrac{34}{3y-5}$.

Since x, y are integers so is $\dfrac{34}{3y-5}$, whence $3y-5 = \pm 1, \pm 2, \pm 17$, or ± 34.

But then y is not an integer unless $3y-5 = 1,\ -2,\ -17$, or 34, which yield the solutions $(x, y) = (5, 2)$, or $(-1, 1)$, or $(-1, -4)$, or $(5, 13)$.

Q.818 If a, b, c are the side lengths of a triangle, A its area, and R the radius of the circumcircle, prove that $abc = 4AR$.



ANSWER Mark Siow (Kogarah Marist High School) writes: Let ABC be any triangle of sides a, b, c. Construct the circumcircle with centre O. Draw the diameter BD.

$B\hat{A}C = B\hat{D}C$ (subtended by the same chord).

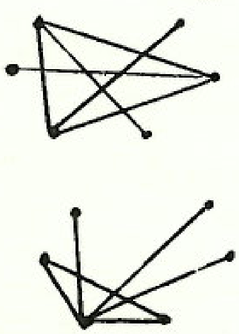
Since BD is a diameter, $B\hat{C}D = 90°$. Therefore

$\sin B\hat{D}C = \dfrac{BC}{BD} = \dfrac{a}{2R}$. Hence $\sin A = \dfrac{a}{2R}$, and $\dfrac{a}{\sin A} = 2R$.

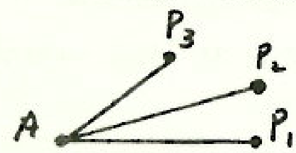
$$\text{Thus } abc = 4(\tfrac{1}{2}bc \sin A)(\frac{a}{2\sin A})$$
$$= 4\times \text{ Area } \times R \text{ (as required).}$$

Also solved by R.R. Rodriguez; A.E. Memorial Science H.S.

Q.819 The figure shows two ways of drawing six line segments joining the vertices of a convex hexagon in each a way that every pair of line segments intersect. It is impossible to draw seven line segments (each an unproduced side or diagonal of the hexagon) with this property. Prove that in fact it is not possible to draw $(n+1)$ sides and/or diagonals of a convex polygon with n vertices in such a way that every pair intersect.



ANSWER The assertion is trivial when $n = 3$; and still obvious when $n = 4$ (any subset of 5 of the 6 sides/diagonals of a quadrilateral must include at least one pair of opposite sides, which do not intersect). If the assertion is false, then there must be a **smallest** number N (clearly greater than 4) such that there exists a convex polygon with N vertices having $N+1$ mutually intersecting sides and/or diagonals. The $N+1$ line segments have $2 \times (N+1)$ ends apportioned amongst the N vertices, so there is at least one vertex A which is an end point of 3 (or more) of the line segments.



Call them AP_1, AP_2 and AP_3 (see Figure). No other line segment can end at P_2, since in that case it could not intersect both AP_1 and AP_3.

Now let us delete the vertex P_2 and the line segment AP_2. We are left with a polygon of $(N-1)$ vertices and N sides and/or diagonals, every pair of which intersect. This contradicts our definition of N. Hence the assertion must be true for all n.

Q.820 Show that, for any positive integer, n, the product of all integers between $n+1$ and $2n$ inclusive is equal to 2^n times the product of all the odd numbers less than $2n$.

ANSWER $(n+1)(n+2)\cdots(2n) = \dfrac{1\ 2\ 3 \cdots n(n+1)\cdots 2n}{1\ 2\ 3 \cdots n}$

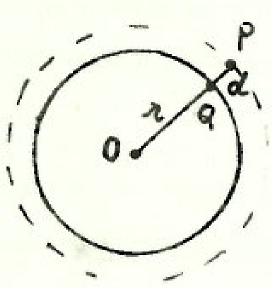
$$= [1\ 3\ 5 \cdots (2n-1) \times \frac{[2\ 4\ 6 \cdots 2k \cdots 2n]}{1\ 2\ 3 \cdots k \cdots n}$$

$$= [1\ 3\ 5 \cdots (2n-1)] \times 2 \times 2 \times 2 \cdots \times 2 \cdots 2$$

$$= 2^n[1\ 3\ 5 \cdots (2n-1)]$$

Q.821 The convex quadrilateral $ABCD$ is not cyclic, and no two sides are parallel. How many circles can be drawn which are equidistant from all four vertices?

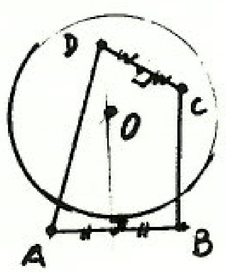
ANSWER Let each of A, B, C, D lie at a distance d from the circle centre O, radius r.



The distance from any point P to the circle is the length PQ, where Q is the point of intersection of the ray OP with the circle. Hence if A, B, C and D are all outside (or all inside) the circle, they all lie on the circle centre O, radius $r+d$ (or radius $r-d$).

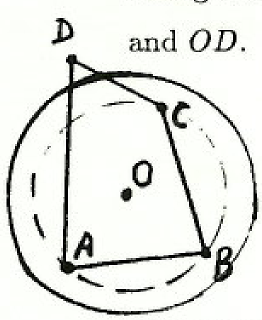
Since the quadrilateral A, B, C, D is not cyclic, this is impossible. Hence the circumference of the circle must partition the vertices $\{A, B, C, D\}$ into two subsets, one set lying outside the circle, the other inside. Either there are two vertices in each of the subsets, or else one subset has three vertices, the other one vertex.

Suppose A, B lie outside the circle, and C, D inside (or else vice versa). Then O lies on the perpendicular bisectors of AB and of CD.



Since AB is not parallel to CD, O is determined uniquely, and r must be taken equal to the arithmetic mean of the length OA and OC. Similarly there is one circle equidistant from A, B, C and D corresponding to the partition $\{A, C\}, \{B, D\}$; and another corresponding to the partition $\{A, D\}, \{B, C\}$.

Now consider the situation where A, B, C are all outside (or all inside) the circle, and D is inside (outside). The centre O must be the circumcentre of the triangle ABC, and the radius r must be the arithmetic mean of the lengths OA and OD.



Similarly there is one circle corresponding to each of the partitions $\{A, B, D\}, \{C\}$; $\{A, C, D\}, \{B\}$; and $\{B, C, D\}, \{A\}$. Thus there are seven different circles equidistant from A, B, C and D.

(Comment: If $ABCD$ is not cyclic, but $AB \parallel CD$ the "circle" having A, B inside and C, D outside degenerates into the straight line parallel to AB and CD and half way between them. Similarly if $ABCD$ is a noncyclic parallellogram, the seven "circles" turn into five circles and two straight lines. Finally if $ABCD$ is cyclic, lying on circle $\mathcal{C}$, every circle concentric with $\mathcal{C}$ is equidistant from all the vertices).

Q.822 If $0 < x_i < 1$ for $i = 1, 2, \cdots n$ and $x_1 x_2 \cdots x_n = (1 - x_1)(1 - x_2) \cdots (1 - x_n)$ find (with proof) the maximum possible value of $P = x_1 x_2 \cdots x_n$.

ANSWER Note that $x_k(1 - x_k) = \frac{1}{4} - (\frac{1}{2} - x_k)^2$ has a maximum value of $\frac{1}{4}$, achieved only when $x_k = \frac{1}{2}$.

Since $x_1 \cdots x_n = (1-x_1)\cdots(1-x_n)$,

$P^2 = (x_1 \cdots x_n)^2 = x_1(1-x_1) \times x_2(1-x_2) \times \cdots \times x_n(1-x_n) \le \frac{1}{4} \times \frac{1}{4} \times \cdots \times \frac{1}{4} = \frac{1}{2^{2n}}$

$\therefore P \le \frac{1}{2^n}$ with equality only when $x_1 = x_2 = x_n = \frac{1}{2}$.

Rico Rodriguez (A.E. Memorial Sc. H.S.) had an alternative idea which I reword as follows:

The geometric mean of the numbers $x_1, x_2, \cdots x_n, (1-x)\cdots(1-x_n)$ is not greater than their arithmetic mean.

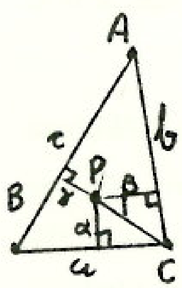
$$\left[\prod_{k=1}^{n} x_k \times \prod_{k=1}^{n}(1-x_k)\right]^{\frac{1}{2n}} \le \frac{\sum_{k=1}^{n} x_k + \sum_{k=1}^{n}(1-x_k)}{2n} = \frac{1}{2}.$$

$$\therefore P^2 \le \frac{1}{2^{2n}};\ P \le \frac{1}{2^n}.$$

Clearly equality is achieved if every $x_k = \frac{1}{2}$.

Q.823 P is a point inside a triangle with sides of length a, b, c. The perpendicular distances from P to these sides are α, β, γ respectively. Prove that $\frac{a}{\alpha} + \frac{b}{\beta} + \frac{c}{\gamma}$ is not less than $\frac{(a+b+c)^2}{2\Delta}$ where Δ is the area of the triangle.

ANSWER Rico Rodriguez gives the following solution:-



$$2\Delta = 2 \times (\Delta PBC + \Delta PCA + \Delta PAB) = a\alpha + b\beta + c\gamma$$

$$\therefore \frac{a}{\alpha} + \frac{b}{\beta} + \frac{c}{\gamma} \ge \frac{(a+b+c)^2}{2\Delta}$$

$$\Leftrightarrow (a\alpha + b\beta + c\gamma)\left(\frac{a}{\alpha} + \frac{b}{\beta} + \frac{c}{\gamma}\right) \ge (a+b+c)^2$$

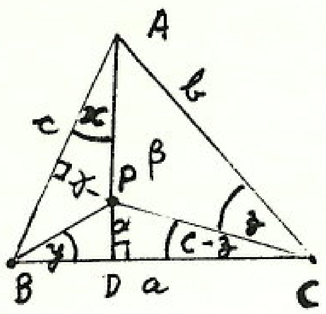
$$\Leftrightarrow ab\left(\frac{\alpha}{\beta} + \frac{\beta}{\alpha}\right) + bc\left(\frac{\beta}{\gamma} + \frac{\gamma}{\beta}\right) + ac\left(\frac{\alpha}{\gamma} + \frac{\alpha}{\alpha}\right) \ge 2ab + 2bc + 2ac \qquad (*)$$

(after removing brackets, subtracting $a^2 + b^2 + c^2$ from both sides, and regrouping).

Since $(\alpha - \beta)^2 \ge 0$, $\alpha^2 + \beta^2 \ge 2\alpha\beta$ whence $ab(\frac{\alpha}{\beta} + \frac{\beta}{\alpha}) \ge 2ab$.

Similarly $bc(\frac{\beta}{\alpha} + \frac{\alpha}{\alpha}) \ge 2bc$; and $ac(\frac{\alpha}{\gamma} + \frac{\gamma}{\alpha}) \ge 2ac$.

Adding these establishes the inequality (*), and hence the equivalent inequality which we were to prove.



Another approach:-

Let $P\hat{A}B = x,\ P\hat{B}C = y,\ P\hat{C}A = z.$

Then $a = BD + DC = \alpha \cot y + \alpha \cot(C - z)$.

$\therefore \dfrac{a}{\alpha} = \cot y + \cot(C - z)$. After similar elucidation of $\dfrac{b}{\beta}$, and $\dfrac{c}{\gamma}$ we have

$$\frac{a}{\alpha} + \frac{b}{\beta} + \frac{c}{\gamma} = [\cot x + \cot(A - x)] + [\cot y + \cot(B - y)] + [\cot z + \cot(C - z)].$$

Either using calculus, on just by elementary trigonometry it is not difficult to show that as x varies from 0 to A the minimum value of $\cot x + \cot(A - x)$ occurs when $x = \dfrac{A}{2}$. Thus the minimum value of $\dfrac{a}{\alpha} + \dfrac{b}{\beta} + \dfrac{c}{\gamma}$ is achieved when P is the point of concurrence of the angle bisectors of the triangle. But then $\alpha = \beta = \gamma = r$ (the radius of the incircle).

and $\min\left(\dfrac{a}{\alpha} + \dfrac{b}{\beta} + \dfrac{c}{\gamma}\right) = \dfrac{a+b+c}{r} = \dfrac{(a+b+c)^2}{2\Delta}$ since $2\Delta = r \times (a + b + c)$.

Q.824 $\{(m - n + 1),\ (m - n + 2), \cdots, (m - 1), m\}$ is a set of n consecutive positive integers with the property that m is a factor of the least common multiple of $\{(m - n + 1), \cdots, (m - 1)\}$. Furthermore, there is no other set of n consecutive positive integers with the same property. Find m and n.

ANSWER Let $M = l.c.m.\{(m - n + 1), (m - n + 2), \cdots, (m - 2), (m - 1)\}$.

When $n = 2, M = m - 1$, and it is impossible that m is a factor of M.

When $n = 3, M = (m - 1) \times (m - 2)$ and again this is never a multiple of m.

[In fact, if m has an odd prime factor p, then the next smaller multiple of p is $m - p$, so p is not a factor of either $(m - 1)$ or $(m - 2)$. Thus M is not a multiple of p and therefore not a multiple of m. If m has no odd prime factor, it is a power of 2 and must be a multiple of 4. But M is not, since $m - 1$ is odd and $(m - 2)$ is the double of an odd number.]

Try $n = 4$. If m contains any prime factor p greater than 3, then none of

$(m-3),(m-2),(m-1)$ has the factor p. Hence M is not a multiple of p, or of m. If 2^2 is a divisor of m, then $(m-1)$ and $(m-3)$ are odd, and $(m-2)$ is the double of an odd number, so M is not a multiple of 4, or of m. Similarly if 3^2 is a divisor of m, M is a multiple of 3, from the factor in $(m-3)$, but not of 3^2, hence not of m. Since $n \geq 4$, there remains only one value of m to be considered; viz, $m = 2 \times 3$. Note that l.c.m. $\{3,4,5\} = 60 = 10 \times 6$. Thus when $n = 4$, there is only one value of $m, m = 6$, such that m is a factor of
l.c.m. $\{(m-n+1), \cdots, (m-2), (m-1)\}$.

We show that $(m,n) = (6,4)$ is the only solution by showing that for any $n \geq 5$ there are always at least two different values of m which are factors of M.

Take $n = 5$. If $m = 6$ then $M =$ l.c.m. $\{2,3,4,5\} = 60 = 10m$.

If $m = 12$ then $M =$ l.c.m. $\{8,9,10,11\} = 3960 = 330m$.

If $n \geq 6$ and 2^e is the largest power of 2 which is less than n (so $\frac{n}{2} \leq 2^e < n$) then m is a factor of M either for $m = 3 \times 2^e$ or for $m = 5 \times 2^e$. (Note that each of these is greater than n; e.g. $3 \times 2^e \geq 3 \times \frac{n}{2} > n$). In fact since $n - 1 \geq 2^e$, those $n-1$ consecutive integers contain at least one number divisible by 2^e, as well as at least one number divisible by 3, and one divisible by 5. Hence M is divisible by $3 \times 5 \times 2^e$. The proof is now complete.

Q.825 Consider all subsets of 8 elements of the set $\{1,2,3,\cdots,17\}$. From each subset select the smallest member. Show that the arithmetic mean of the 24,310 numbers selected is equal to 2.

ANSWER If the least element in a subset is k, the remaining 7 elements in the subset all lie in $\{k+1, k+2, \cdots, 17\}$. Hence there are ${}^{17-k}C_7$ of the subsets with least element equal to k, for $k = 1, 2, \cdots 10$. Thus the arithmetic mean of the least elements is

$$\frac{1}{{}^{17}C_8}(1 \times^{16} C_7 + 2 \times^{15} C_7 + 3 \times^{14} C_7 + \cdots + k \times^{17-k} C_7 + \cdots + 10^7 C_7)$$

Of course, it is easy if a little tedious to calculate the answer 2 from here. Instead we shall simplify the expression in parentheses until the numerical calculation is very short. Remember that the Pascal triangle property of the binomial coefficients is ${}^NC_R = {}^{N-1}C_{R-1} + {}^{N-1}C_R$. Using this

$$1 \times {}^{16}C_7 + 2 \times {}^{15}C_7 + 3 \times {}^{14}C_7 + \cdots + 9 \times {}^{8}C_7 + 10 \times {}^{7}C_7$$

$$= 1 \times ({}^{17}C_8 - {}^{16}C_8) + 2 \times ({}^{16}C_8 - {}^{15}C_8) + 3 \times ({}^{15}C_8 - {}^{14}C_8)$$

$$+ \cdots + 9 \times ({}^{9}C_8 - {}^{8}C_8) + 10 \times ({}^{8}C_8 - 0)$$

$$= {}^{17}C_8 + {}^{16}C_8 + {}^{15}C_8 + \cdots + {}^{8}C_8$$

$$= ({}^{18}C_9 - {}^{17}C_9) + ({}^{17}C_9 - {}^{16}C_9) + ({}^{16}C_9 - {}^{15}C_9) + \cdots + ({}^{9}C_9 - 0)$$

$$= {}^{18}C_9.$$

Therefore the average required is $\dfrac{{}^{18}C_9}{{}^{17}C_8} = \dfrac{18!}{9!9!}\dfrac{8!9!}{17!}$

$$= \frac{18}{9} = 2.$$

(In fact if 17 is replaced by n in this question and 8 is replaced by r, the above working produces the answer for the average of the least members $\dfrac{{}^{n+1}C_{r+1}}{{}^{n}C_r} = \dfrac{n+1}{r+1}$).

Correct solution from M. Siow (Kogarah Marist High School).

Q.826 The Fibonacci numbers are $\{1, 2, 3, 5, 8, 13, 21, \cdots\}$ where each number after the second is the sum of the previous two.

(i) Prove that every positive integer can be expressed as a sum of (one or more) distinct Fibonacci numbers.

(For example 21 can be so expressed in four different ways:- $21 = 21$; $21 = 13 + 8$; $21 = 13 + 5 + 3$; $21 = 13 + 5 + 2 + 1$).

(ii) Find, with proof, all positive integers such that there is only one such expression.

(e.g. $33 = 21 + 8 + 3 + 1$).

ANSWER I shall denote the kth number in the given list by $F_k, k = 1, 2, \cdots$. Furthermore I shall use $\mathcal{F}$ as an abbreviation of "Fibonacci number".

(i) The assertion is easily checked for small positive integers. Suppose there are positive integers which cannot be expressed as the sum of distinct $\mathcal{F}$s, and let x be the smallest of them. For some $n, F_{n-1} < x \leq F_n$. Let $y = x - F_{n-1}$. Then $0 < y \leq F_n - F_{n-1} = F_{n-2}$. Since y is a positive integer less than x, it is expressible as the sum of distinct $\mathcal{F}$s, and since it does not exceed F_{n-2}, none of the $\mathcal{F}$s used can be equal to F_{n-1}. Hence $x = y + F_{n-1}$ can also be expressed as the sum of distinct $\mathcal{F}$s. But this contradicts our definition of x. We conclude that there cannot exist any positive integer not expressible in the stated form.

(ii) We prove two preliminary results.

PR.1 $\sum_{k=1}^{n} F_k = F_{n+2} - 2$ for $n = 1, 2, 3, \cdots$

[Proof by mathematical induction. It is true when $n = 1$ since $F_1 = 1 = F_3 - 2$. Assuming it is true when $n = m$, (i.e. $\sum_{k=1}^{m} F_k = F_{m+2} - 2$), consider $\sum_{k=1}^{m+1} F_k$.

This is $(F_{m+2} - 2) + F_{m+1} = F_{m+3} - 2$. Thus the assertion is still true for $n = m + 1$. Hence it is true for all natural numbers n.]

PR.2 Every positive whole number not exceeding $F_{n+2} - 2$ is expressible as a sum of distinct $\mathcal{F}$s selected from $\{F_1, F_2, \cdots, F_n\}$.

Proof. This can be checked for small numbers n. Suppose the assertion is false. Then there is a **smallest** number N, and a whole number x not exceeding $F_{N+2} - 2$ which is **not** expressible as the sum of a subset of $\{F_1, F_2, \cdots, F_N\}$. Obviously x must be greater than F_N (because of (i)). Let $y = x - F_N$. Then $y \leq F_{N+2} - 2 - F_N = F_{N+1} - 2$.

By our definition of N, y **is** expressible as the sum of distinct $\mathcal{F}$s selected from $\{F_1, F_2, \cdots, F_{N-1}\}$. Hence $x = F_N + y$ is expressible as the sum of distinct $\mathcal{F}$s chosen from $\{F_1, F_2, \cdots, F_N\}$ contradicting the definition of x. Thus the assertion cannot be false.

We are now ready to answer the question asked.

We claim that a positive whole number x has a **unique** decomposition into distinct $\mathcal{F}$s if and only if $x + 1$ is a $\mathcal{F}$. Check this for small values of x.

Proof IF. Let $x = F_{n+2} - 1$. Because of PR.1, x is too large to be expressed as the sum of distinct $\mathcal{F}$s not exceeding F_n. Hence F_{n+1} must occur in any such expression for x. But then $x - F_{n+1} = F_{n+2} - 1 - F_{n+1} = F_n - 1$ which is one less than a smaller $\mathcal{F}$. Repeating the process we see that the **only** such representation of $F_{n+2} - 1$ is $F_{n+1} + F_{n-1} + F_{n-3} + \cdots + F_k$ where k is 2 or 1 according as n is odd or even.

ONLY IF. We need to show that if $F_{n+1} \leq x \leq F_{n+2} - 2$ there are at least two different representations of x in the stated form. This is obvious if $F_{n+1} = x$ (viz. $x = F_{n+1} = F_n + F_{n-1}$). If $F_{n+1} < x$, then there is at least one representation of x using F_{n+1} as one of the terms, since $z(= x - F_{n+1})$ is less than F_n and by (1) is expressible as the sum of distinct $\mathcal{F}$s. By PR.2 there is a second representation not using F_{n+1} as one of the terms. This completes the proof.

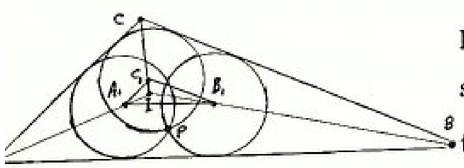


fig 1

Q.827 In the diagram three circles of equal size all pass through a point P. The triangle $\triangle ABC$ encloses the three circles, each side being tangential to two of them. Let I be the point inside the triangle equidistant from the three sides, this distance being r, and let O be the circumcentre of $\triangle ABC$, with $OA = OB = OC = R$. Prove that P lies on OI, and that $\dfrac{OP}{PI} = \dfrac{R}{r}$.

ANSWER Label the centres of the three equal circles A_1, B_1, C_1 (see figure 1), and denote the length of their radii by d. We make the following observations.

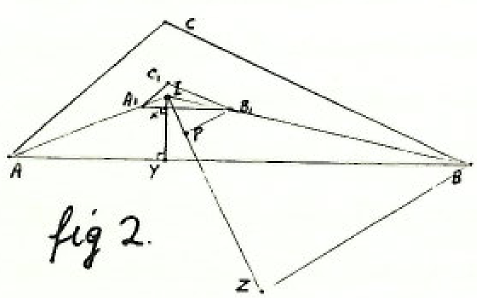


fig 2.

(i) Since the perpendicular distances of A_1 from AB and from AC are each equal to d, AA_1 is the bisector of $\widehat{A}$. Similarly BB_1 and CC_1 are bisectors of $\widehat{B}$ and $\widehat{C}$, so AA_1, BB_1, and CC_1 produced are concurrent at the incentre, I.

(ii) Since A_1 and C_1 are equidistant from AC, we have $A_1C_1 \parallel AC$.

Similarly $A_1B_1 \parallel AB$, and $B_1C_1 \parallel BC$. Hence the

$\triangle A_1B_1C_1$ is similar to $\triangle ABC$, and they have the same incentre I, since AA_1 produced is easily proved to bisect $\widehat{A}_1$.

Let the magnification factor $\dfrac{AB}{A_1B_1} = \dfrac{IA}{IA_1}$ be denoted by λ. If IXY is perpendicular to A_1B_1 and AB, then since $\triangle AYI \equiv \triangle A_1XI$

$$\lambda = \frac{AI}{A_1I} = \frac{YI}{XI} = \frac{r}{r-d} \quad \text{(see figure 2)} \qquad (*)$$

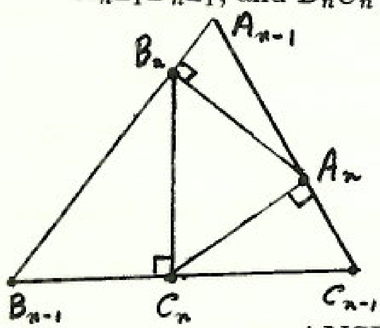
(iii) Since $A_1P = B_1P = C_1P = d$, P is the circumcentre of $\triangle A_1B_1C_1$. Let Z be the point on IP produced with $\dfrac{IZ}{IP} = \lambda$. Then $\triangle IB_1P \equiv \triangle IBZ$ (check thus) and $\dfrac{B_1P}{BZ} = \dfrac{IB_1}{IB} = \dfrac{1}{\lambda}$; whence $BZ = \lambda B_1P = \lambda d$. Similarly $CZ = \lambda d$ and $AZ = \lambda d$. $\therefore$ Z coincides with O, the circumcentre of $\triangle ABC$, and $R = OA = ZA = \lambda d$.

From $(*)$, $\lambda r - \lambda d = r \Rightarrow \lambda r - R = r \Rightarrow \lambda = \frac{r+R}{r}$

Finally $\dfrac{OP}{PI} = \dfrac{OI-PI}{PI} = \dfrac{OI}{PI} - 1 = \lambda - 1 = \dfrac{r+R}{r} - \dfrac{r}{r} = \dfrac{R}{r}$.

Thus everything has been proved.

Q.828 Let $\triangle A_1B_1C_1$ be any given triangle. For $n = 2, 3, 4, \cdots$ let $\triangle A_nB_nC_n$ be the triangle inscribed in $\triangle A_{n-1}B_{n-1}C_{n-1}$ such that $C_nA_n \perp C_{n-1}A_{n-1}$, $A_nB_n \perp A_{n-1}B_{n-1}$, and $B_nC_n \perp B_{n-1}C_{n-1}$ (see figure).



(i) Show how to construct with ruler and compass the triangle $\triangle A_2B_2C_2$.

(ii) Show that there is a point P (independent of n) which lies on the circles having diameters $A_{n-1}A_n$, $B_{n-1}B_n$, and $C_{n-1}C_n$.

ANSWER (i) Let X be any point on A_1C_1 .

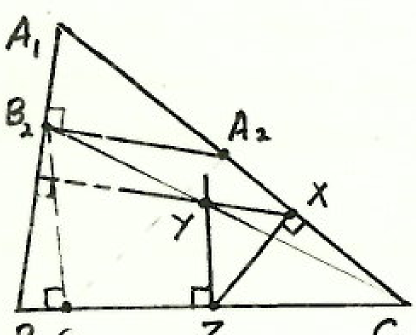


fig 1

Construct $XZ \perp A_1C_1$ intersecting B_1C_1 at Z. Construct at Z a line perpendicular to B_1C_1, and through X a line perpendicular to A_1B_1. Let these lines intersect at Y. Let C_1Y (produced if necessary) intersect A_1B_1 at B_2. Construct B_2A_2 perpendicular to A_1B_1 intersecting A_1C_1 at A_2, and B_2C_2 perpendicular to B_1C_1 intersecting B_1C_1 at C_2.

Proof. It only remains to prove that $C_2A_2 \perp A_1C_1$.

Since

$$B_2A_2 \parallel YX, \; \frac{C_1A_2}{C_1X} = \frac{C_1B_2}{C_1Y} = \frac{C_1C_2}{C_1Z} (\text{since } B_2C_2 \parallel YZ)$$

$\therefore C_2A_2 \parallel ZX$

$\therefore C_2A_2 \perp A_1C_1$.

(ii) We shall denote the angles of $\triangle A_kB_kC_k$ by $\widehat{A}_k, \widehat{B}_k, \widehat{C}_k$.

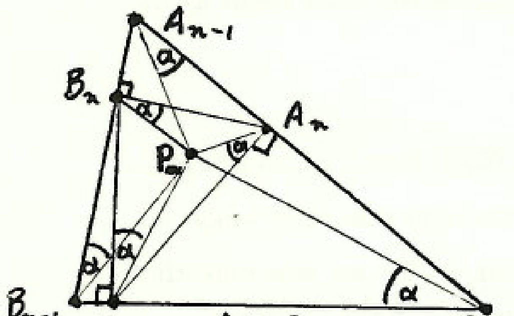


fig 2

Since $A_{n-1}\widehat{A}_nB_n$ is complementary to both $\widehat{A}_{n-1}$ and $\widehat{A}_n$, $\widehat{A}_{n-1} = \widehat{A}_n$. Similarly $\widehat{B}_{n-1} = \widehat{B}_n$ and $\widehat{C}_{n-1} = \widehat{C}_n$, so $\triangle A_{n-1}B_{n-1}C_{n-1} \equiv \triangle A_nB_nC_n$.

Let the circles on diameters $A_{n-1}A_n$ and $B_{n-1}B_n$ intersect at P_n (as well as at B_n). Since $A_{n-1}B_nP_nA_n$ and $B_{n-1}C_nP_nB_n$ are both cyclic quadrilaterals $B_n\widehat{P}_nA_n = 180° - \widehat{A}_{n-1}$ and $C_n\widehat{P}_nB_n = 180° - \widehat{B}_{n-1}$.

Hence $C_n\widehat{P}_nA_n = 360° - (180° - \widehat{A}_{n-1}) - (180° - \widehat{B}_{n-1}) = \widehat{A}_{n-1} + \widehat{B}_{n-1} = 180° - \widehat{C}_{n-1}$. $\therefore C_nC_{n-1}A_nP_n$ is also a cyclic quadrilateral, and since $C_n\widehat{A}_nC_{n-1} = 90°$, C_nC_{n-1} is a diameter.

All that remains to be shown is that P_n is independent of n. Note that

$$\begin{aligned} P_n\widehat{A}_nC_n &= P_n\widehat{A}_{n-1}C_{n-1} \quad (\text{both complementaryto } P_n\widehat{A}_nA_{n-1} \\ &= P_n\widehat{B}_nA_n \quad (\text{from cyclic quadrilateral } P_nA_nA_{n-1}B_n) \\ &= P_n\widehat{B}_{n-1}A_{n-1} \quad (\text{both complementary to } P_n\widehat{B}_nB_{n-1}) \\ &= P_n\widehat{C}_nB_n = P_n\widehat{C}_{n-1}B_{n-1} \quad (\text{similar reasons}) \end{aligned}$$

(These angles are labelled α in fig.2)

It is evident that the size of these angles is determined by the shape of $\triangle A_{n-1}B_{n-1}C_{n-1}$ (In fact, one expression for α is

$$\cot\alpha = \frac{\sin^2\widehat{A}_{n-1} + \cos\widehat{A}_{n-1}\sin\widehat{B}_{n-1}\sin\widehat{C}_{n-1}}{\sin\widehat{A}_{n-1}\sin\widehat{B}_{n-1}\sin\widehat{C}_{n-1}},$$

but we shall omit its derivation since the actual formula is not essential for the argument).

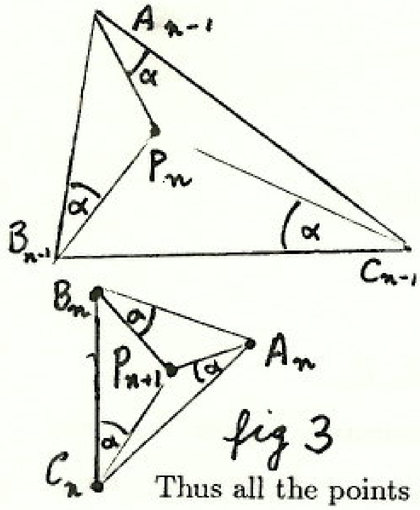


fig 3

Now when $\triangle A_{n+1}B_{n+1}C_{n+1}$ is obtained from $\triangle A_nB_nC_n$, identical working to the preceding shows that circles on diameters A_nA_{n+1}, B_nB_{n+1}, and C_nC_{n+1} are concurrent at a point P_{n+1}, such that the angles $P_{n+1}\widehat{A}_nC_n$, $P_{n+1}\widehat{C}_n$ $P_{n+1}\widehat{B}_nA_n$ are all equal. Furthermore since $\triangle A_nB_nC_n$ is the same shape as $\triangle A_{n-1}B_{n-1}C_{n-1}$, those equal angles are all equal to α (see fig.3). Comparing this with fig.2 it is now plain that P_{n+1} is the same point as P_n. Thus all the points P_n, $n = 1, 2, \cdots$ coincide.

This completes the proof.

(Calling this point P, the operation of obtaining $\triangle A_nB_nC_n$ from $\triangle A_{n-1}B_{n-1}C_{n-1}$ can be described as a rotation about the point P through one right angle followed by a reduction of lengths by the factor $\lambda = \tan\alpha$. To see this note that $\triangle A_{n-1}PC_{n-1} \equiv \triangle A_nPC_n$ and deduce $\lambda = \dfrac{A_nC_n}{A_{n-1}C_{n-1}} = \dfrac{PA_n}{PA_{n-1}} = \tan\alpha$, etc.)

Late Solutions:

We received good solutions of Q805, Q808 and Q815 from Rico R. Rodriguez (Arturo Eustoquio Memorial H.S. Philippines) too late for acknowledgement in Volume 26, No.3.