JUNIOR

1. Alice's and Bert's ages combined total 11016 days. In another 1296 days Bert will be twice as old as Alice was when Bert was twice as old as Alice was when Alice was twice as old as Bert.
How old is each now?

ANS. Let the difference in their ages be x days. When Alice was twice as old as Bert their ages were $2x$ days and x days. When Bert's age was $2 \times (2x)$ days, Alice's age was $5x$ days. Therefore in 1296 days time Bert's age will be $2 \times (5x)$ days and Alice's age will be $11x$ days.

$$\therefore (10x - 1296) + (11x - 1296) = 11016.$$

Solving this yields $x = 648$days.
So Alice's age is $11x - 1296 = 11 \times 648 - 1296 = 5832$ days and Bert's age is $10 \times 648 - 1296 = 5184$ days.

2. 1991 people, no two of exactly the same height, are standing in a ring. Prove that there are four people A, B, C, D standing together (B, C, D on the immediate right of A, B, C respectively) such that A is shorter than C, but B is taller than D.

ANS. EITHER Label the people $P_i; i = 1, 2, \cdots, 1991$ in order around the ring, and let h_i denote the height of P_i for $1 \leq i \leq 1991$, and $h_{1992} = h_1, h_{1993} = h_2, h_{1994} = h_3$.
Note that $\sum_{i=1}^{1991}(h_{i+2} - h_i) = (h_3 - h_1) + (h_4 - h_2) + (h_5 - h_3) + \cdots + (h_{1991} - h_{1989} + (h_1 - h_{1990}) + (h_2 - h_{1991}) = 0$, since on removing brackets every h_i occurs twice, with opposite signs. Hence it is impossible that all terms have the same sign. Either we can find two neighbouring terms in the sum the first positive and the second negative (no term is zero), or the final term is positive and the initial term is negative. That is we can find k in $\{1, 2, \cdots 1991\}$ such that $h_{k+2} > h_k$ but $h_{k+3} < h_{k+1}$. Then $A = P_k, B + P_{k+1}, C = P_{k+2}, D = P_{k+3}$ are such that A is shorter than C, but B is taller than D.

OR Essentially the same argument results from the observation that

$$h_1 < h_3 < h_5 < \cdots < h_{1991} < h_2 < h_4 \cdots < h_{1990} < h_1$$

is impossible, since $h_1 \not< h_1$. (i.e. heights cannot <u>always</u> increase when you go two places to the right); and likewise it is impossible that $h_1 > h_3 > \cdots >$

$h_{1991} > h_2 > \cdots > h_{1990} > h_1$. So $h_{i+2} - h_i$ is sometimes positive and sometimes negative. Now proceed as before.

OR Let X be the tallest person and consider the group $VW\ X\ YZ$. If W is taller than Y we may take $A = V, B = W, C = X, D = Y$.
If W is shorter than Y, take $A = W, B = X, C = Y$ and $D = Z$.

3. When the bank teller cashed my cheque she accidentally interchanged the dollars and cents, so giving me just 22 cents less than three times the correct amount. What sum did I need to return to correct the error?

ANS. If the cheque was for $\$x$ and y cents, since three times its value is still not over \$100.22, $x < 34$. From the data, the amount proffered was $(100y + x)$ cents and this is $[3 \times (100x + y) - 22]$ cents.

$$\therefore 100y + x = 3 \times 100x + 3y - 22$$
$$97y = 299x - 22$$
$$97(y - 3x) = 8x - 22 \ < 8 \times 34 - 22 = 250.$$

The only even multiple of 97 less than 250 is 97×2.

$$\therefore y - 3x = 2 \text{ and } 97 \times 2 = 8x - 22.$$

Solving yields $x = 27, y = 83$ and the amount to be returned is \$83.27 - \$27.83 = \$55.44.

4. Find all positive integers n such that

$$2^n + 65 \text{ is a perfect square.}$$

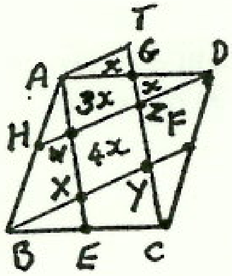
Prove your assertion.

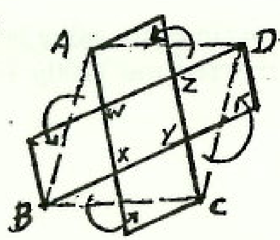
ANS. When n is an odd integer, the last digit of 2^n is always either 2 or 8 (since if M ends with 2, $4 \times M$ ends with 8 and vice versa). Hence the last digit of $2^n + 65$ is either 7 or 3.
Since whatever the last digit of x, x^2 ends with one of 0, 1, 4, 9, 6, or 5, we conclude that $2^n + 65$ is never a perfect square when n is an odd integer.
When n is even, $n = 2m$ for some positive integer m.
If $2^{2m} + 65 = x^2$, then $x^2 - (2^m)^2 = 65$

$$(x - 2^m)(x + 2^m) = 65 = 1 \times 65 = 5 \times 13.$$

Either $x - 2^m = 1$ and $x + 2^m = 65$ (1)
or $x - 2^m = 5$ and $x + 2^m = 13$. (2)
The solution of (1) is $2x = 66, x = 33, 2^m = 32, m = 5$.
The solution of (2) is $2x = 18, x = 9, 2^m = 4, m = 2$.
Hence $n = 2m = 2 \times 5$ or 2×2.
The only positive integers n such that $2^n + 65$ is a perfect square are $n = 10$ or 4.

5. $ABCD$ is a parallelogram of unit area and E, F, G, H are mid-points of the sides BC, CD, DA, AB respectively. The line segments AE, BF, CG and DH dissect the interior of $ABCD$ into nine regions. Find the area of the central region.





ANS. It is not difficult to prove that the central figure $WXYZ$ is another parallelogram. In fact, since $AG = \frac{1}{2}AD = \frac{1}{2}BC = EC$, and $AG \parallel EC$, $AECG$ is a parallellogram.
$\therefore AE \parallel GC$. Similarly $DH \parallel BE$.

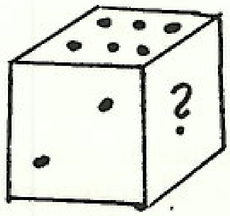
Since G is the mid point of DA and $GC \parallel AE$, Z is the mid point of DW. It follows that if area $\triangle DGZ$ is x then area $\triangle DAW = 4x$ (when the mid points of the sides of a triangle are joined, it is dissected into 4 smaller congruent triangles). Thus area $AWZG = 4x - x = 3x$. Construct $AT \parallel WZ$. Since $\triangle AGT \equiv \triangle DGZ$, area $AWZT = 4x$. Since W is the mid point of AX, the parallelogram $WXYZ$ is congruent to $AWZT$, so has area $4x$. Similarly, area $WXYZ$ is four times the area of any of the other corner triangles, so they all have area x, and the remaining edge regions all have area $3x$. Thus area $ABCD = 4 \times x + 4 \times 3x + 4x = 20x = 1$, whence $x = \frac{1}{20}$ and finally area $WXYZ = 4x = \frac{1}{5}$.
(By rotating the corner triangles through a half turn about E, F, G or H one can reassemble the nine regions into 5 congruent parallellograms as shown in the second figure.)

6. A "cubical die" is a cube with the numbers 1,2,3,4,5 and 6 on the faces.
If the sum of the numbers on each pair of opposite faces is 7, it is a "standard cubical die". Two dice are to be regarded as "the same" if they can be placed so that the same numbers are visible on both dice from every direction.

i) How many different standard cubical dice are there?

ii) How many different cubical dice are there?

iii) How many different cubical dice are there such that **no** pair of opposite faces have numbers which add to 7?



ANS. (i)Place the die on the table with number 6 uppermost, (so 1 is in contact with the table) and turn it so that the front face has the number 2. Then the back face must have the number 5. The remaining numbers 3 and 4 are on the side faces. There are 2 possibilities, depending on which number is on the right hand face. Thus there are 2 different standard cubical dice.

(ii) Place the die with the number 1 in contact with the table. There are 5 different possibilities for the number on the uppermost face. Whichever actually applies, we shall show that there are 6 different dice having that number opposite 1, so that there are altogether $5 \times 6 = 30$ different dice.

For definiteness suppose that the uppermost face carries the number 5. We can turn the die so that the front face has the number 2. There remain three possibilities for the back face $\{3, 4, \text{ or } 6\}$ and whichever actually applies there are two different possibilities for the remaining two numbers on the right and left faces. Hence, there are three times two or six different dice with 1 and 5 on opposite faces.

(iii) There are 16 different dice if no pair of opposite numbers add to 7.

As before place the die with 1 in contact with the table. Since 6 is not uppermost we can turn the die so that the front face shows 6.

There are four possibilities for the uppermost face. We shall show that whichever of these applies, there are four different ways in which the remaining three numbers could be placed, thus establishing the result.

For definiteness, suppose 5 is uppermost. The number $2(= 7 - 5)$ cannot be opposite 6 since that would force the remaining numbers 3 and 4 to be on opposite faces. So there are only two possibilities for the number on the back face (opposite 6), and for either of these there are two different ways to distribute 2 and the remaining number on the right and left faces.

SENIOR

1. 1991 people, no two of exactly the same height, are standing in a ring. Prove that there are four people A, B, C, D standing together (B, C, D on the immediate right of A, B, C respectively) such that A is shorter than C, but B is taller than D.

ANS. See question 2 on the Junior paper.

2. The positive numbers $a_1, a_2, \cdots, a_n$ form an arithmetic progression. Prove that

$$\frac{1}{\sqrt{a_1}+\sqrt{a_2}}+\frac{1}{\sqrt{a_2}+\sqrt{a_3}}+\cdots+\frac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}}=\frac{n-1}{\sqrt{a_1}+\sqrt{a_n}}.$$

ANS. Since $\frac{1}{\sqrt{x}+\sqrt{y}} = \frac{\sqrt{y}-\sqrt{x}}{y-x}$ the left hand side of the equation is equal to

$$\frac{(\sqrt{a_2}-\sqrt{a_1})}{d}+\frac{(\sqrt{a_3}-\sqrt{a_2}}{d}+\cdots+\frac{(\sqrt{a_n}-\sqrt{a_{n-2}})}{d}$$

(where $d = a_k - a_{k-1}$ is the common difference of terms in the arithmetic progression.)

$$=\frac{1}{d}(\sqrt{a_n}-\sqrt{a_1})=\frac{1}{d}\frac{a_n-a_1}{\sqrt{a_n}+\sqrt{a_1}}$$

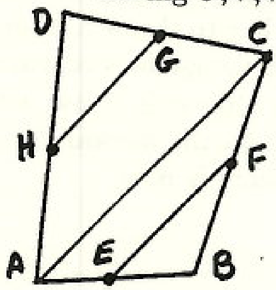
$$=\frac{1}{d}\frac{(n-1)d}{(\sqrt{a_1}+\sqrt{a_n})}=\frac{n-1}{\sqrt{a_1}+\sqrt{a_n}}.$$

3. Find all positive integers n such that $2^n + 65$ is a perfect square. Prove your assertion.

ANS. See question 4 on the Junior paper.

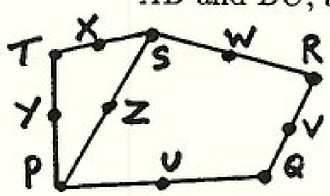
4. i) Given three non-collinear points E, F, G construct a fourth point H such that there can be found a quadrilateral $ABCD$ having E, F, G, H as the mid-points of AB, BC, CD, DA respectively.

ii) Given a convex pentagon $UVWXY$, show how to construct five points P, Q, R, S, T having U, V, W, X, Y as mid-points of PQ, QR, RS, ST and TP respectively.



ANS. i) Analysis: If $ABCD$ is such a quadrilateral, then since E, F are midpoints of the sides AB and BC in $\Delta ABC, EF \parallel AC$ and $EF = \frac{1}{2}AC$.
Similarly $GH \parallel AC$ and $GH = \frac{1}{2}AC$ (from ΔDAC).
$\therefore EFGH$ is a parallelogram.

Construction: Construct a line through E parallel to FG and a line through G parallel to FE. Their point of intersection is point H.
(Proof: Clearly $EFGH$ is a parallelogram. Construct the line through the mid point of FG parallel to EF and let AC be any interval on this line of length $2 \times EF (= 2 \times HG)$. Let AH and CG intersect at D, and AE and CF intersect at B. (See figure.)
It is very easy to prove $\Delta BEF \sim \Delta BAC$ and deduce that E, F are midpoints of AB and BC, and similarly for the other half of the figure.



ii) Analysis: Suppose $PQRST$ are the correct points and let Z be the mid point of SP.
By (i) $UVWZ$ is a parallelogram (and we can construct Z). Now in $\Delta PST, XY \parallel PS$ and $PZ = YX = ZS$.

Hence construction: construct Z as the point of intersection of a line through U parallel to VW with a line through W parallel to VU.
Through Z construct a line parallel to YX and construct on it points P, S such that $PZ = YX = ZS$. Let PY and SX intersect at T. Double PU at Q, and SW at R.
Proof. Since $\Delta TXY \sim \Delta TSP$ (because $YX \parallel PS$) and $XY = \frac{1}{2}PS, X$ and Y are mid points of ST and TP. Let the mid point of QR be V'. Then, as in (i), $WZUV'$ is a parallelogram. Hence V' coincides with V.

5. i) I throw a die (with the usual six sides) once. I then have the choice of taking the number thrown, or throwing again and taking the second number. My aim is to obtain as large a number as possible. What is my best strategy? If I repeat this game many times with the best strategy what can I expect to be the average of my scores?

ii) As in (i), except that I am allowed three throws. I may stop after the first or second throw if I wish, and take the number I have just thrown.
What is my expected average score using the best strategy?

iii) If up to n throws are allowed, and I may stop after any throw if I wish, what strategy should I employ, and what then is my expected average score?

ANS. i) The expected average for one throw of the die is $\frac{1}{6}(1+2+3+4+5+6) = 3\frac{1}{2}$, since each of the possible scores $1, 2, 3, 4, 5, 6$ are equally likely. Hence my best strategy is to take the first number thrown if it is 4,5 or 6, but to throw a second time if the first throw is less than $3\frac{1}{2}$. With this strategy, if the game is repeated N times (N large), I will expect about $\frac{N}{2}$ games to stop after one throw with average score $\frac{4+5+6}{3} = 5$, and the remaining games to go to the second throw with an average score of $3\frac{1}{2}$. Hence my expected average score is now

$$\frac{1}{N}\left(\frac{N}{2} \times 5 + \frac{N}{2} \times 3\frac{1}{2}\right) = 4\frac{1}{4}.$$

ii) If the first throw is less than $4\frac{1}{4}$, I should reject it, and then use the strategy in (i) for the remaining two throws. If the first throw is 5 or 6, since it is greater than $4\frac{1}{4}$, I should take that score. Since a 5 or 6 is thrown in about $\frac{N}{3}$ throws my expected average score with this strategy is

$$\frac{1}{N}\left(\frac{N}{3} \times \frac{5+6}{2} + \frac{2N}{3} \times 4\frac{1}{4}\right) = 4\frac{2}{3}.$$

iii) If 4 throws are permitted, on the first throw I should reject a score $< 4\frac{2}{3}$, but accept a score $> 4\frac{2}{3}$. The expected average score will be

$$\frac{1}{N}\left(\frac{N}{3} \times 5\frac{1}{2} + \frac{2N}{3} \times 4\frac{2}{3}\right) = 4\frac{17}{18}.$$

Since $4\frac{17}{18} < 5$, thé same strategy is employed on the first throw if $n = 5$. (i.e. accept 5 or 6, reject 1, 2, 3, or 4) and the expected average score is $\frac{1}{3} \times 5\frac{1}{2} + \frac{2}{3} \times 4\frac{17}{18} = 5\frac{7}{54}$.
Since this exceeds 5 the best strategy on the first throw when n is six or more is to reject any throw except a 6, and persist with this strategy until only five throws remain. Then proceed as above.

Let s_n be the expected score for the n-throw game. Then for $n \geq 5$

$$s_{n+1} = \frac{1}{N}\left(\frac{N}{6} \times 6 + \frac{5N}{6} \times s_n\right)$$

$$= 1 + \frac{5}{6}s_n, \text{ which is the same as}$$

$$6 - s_{n+1} = \frac{5}{6}(6 - s_n). \qquad (*)$$

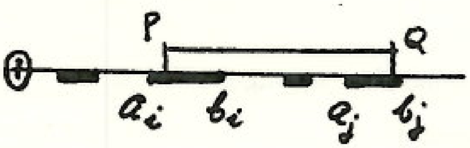
$$\text{Since } \quad 6 - s_5 = \frac{47}{54}, (*) \text{ gives } 6 - s_6 = \frac{5}{6} \times \frac{47}{54},$$

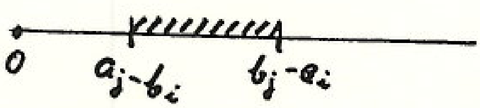
$$\text{then } \quad 6 - s_7 = \frac{5}{6}(6 - s_6) = (\frac{5}{6})^2 \times \frac{47}{54} \text{ and so on.}$$

$$\text{In fact } \quad 6 - s_n = (\frac{5}{6})^{n-5} \times \frac{47}{54} \text{ for any } n > 5$$

$$\text{and } \quad s_n = 6 - (\frac{5}{6})^{n-5} \times \frac{47}{54}.$$

6. A set S consists of k non-overlapping line segments all lying on a straight line. The sum of their lengths is L. Any line segment of length less than D can be placed on the straight line so that both its end points are in S. Prove that L is not less than D/k.





ANS. Choose an origin, 0 on the straight line, and let the ith segment have end points a_i, b_i units of length to the right of 0. (So, since the line segments do not overlap $a_1 < b_1 < a_2 < b_2 < \cdots < a_k < b_k$). Let $b_i - a_i = \ell_i$, the length of the ith segment.

We are given $\sum_{i=1}^{k} \ell_i = L$.

Consider a segment PQ whose end points lie in (a_i, b_i) and (a_j, b_j), where $i < j$. The length of PQ can be anything between a minimum of $a_j - b_i$ up to a maximum of $b_j - a_i$.

On a second number axis we shade the region from $a_j - b_i$ to $b_j - a_i$ to indicate that a segment whose length lies in the shaded region can be placed on S as required.

The length of the interval shaded is $(b_j - a_i) - (a_j - b_i) = (b_i - a_i) + (b_j - a_j) = \ell_i + \ell_j$. If we repeat this process for all possible pairs of segments, the sum of the lengths of shaded intervals is

$$\sum_{1 \le i < j \le k} (\ell_i + \ell_j) = (\ell_1 + \ell_2) + (\ell_1 + \ell_3) + \cdots + (\ell_1 + \ell_k)$$
$$+ (\ell_2 + \ell_3) + (\ell_2 + \ell_4) + \cdots + (\ell_2 + \ell_k) + \cdots + (\ell_{k-1} + \ell_k).$$

Each of the segments is paired with the other $(k-1)$ in this sum, so ℓ_i occurs $(k-1)$ times on the RHS for every i. Hence the sum of the lengths of shaded intervals is $(k-1)(\ell_1 + \ell_2 + \cdots + \ell_k) = (k-1)L$. Of course, if the segment PQ has length less than the longest segment (a_i, b_i) it can be placed in that interval. So we also shade the region on the second number axis between 0 and max ℓ_i. This shaded interval clearly has length $\le L$ so that our combined sum of lengths shaded is now $\le (k-1)L + L = kL$.

Now we are told that any line segment of length less than D can be placed on S, so the shaded region includes all numbers on the second axis from 0 to D. Hence $D \le kL$, or $L \ge D/k$. (In fact; if the shaded intervals overlap anywhere, or if the longest segment (a_i, b_i) has length $< L$, we would have $D < kL$. Equality can be achieved only if one of the segments is of length L and the others are all single point "segments" equally spaced at distances of L.)