Parabola Volume 27, Issue 2 (1991)

PROBLEM SECTION

You are invited to submit solutions to any or all of the following problems, accompanied by your name, school and year or form. Solutions of these problems will appear in the next issue of Parabola; your solution(s) may be used if they are received in time.

Q.840 (i) Let α, β be two distinct solutions of

$$x^3 - x^2 - x + c = 0$$

Simplify $\alpha^2\beta + \alpha\beta^2 - \alpha\beta$.

(ii) Let α, β be two distinct solutions of

$$x^4 + x^3 + kx^2 - x - 1 = 0.$$

Simplify $\alpha^3\beta^2 + \alpha^2\beta^3 + \alpha^2\beta^2 + \alpha\beta + \alpha + \beta$.

Q.841 None of a, b, c is zero, and α, β are the roots of $ax^2 + bx - c = 0$. If $2\alpha, 2\beta$ are the roots of $a^2x^2 + b^2x - c^2 = 0$ find the roots of $a^3x^2 + b^3x - c^3 = 0$.

Q.842 Prove that if for some non-negative number c

$$\frac{a_1}{c+1} + \frac{a_2}{c+2} + \frac{a_3}{c+3} + \cdots + \frac{a_n}{c+n} = 0$$

then the equation

$$a_1 + a_2x + a_3x^2 + \cdots + a_nx^{n-1} = 0$$

has a solution x lying between 0 and 1.

Q.843 The numbers a, b, c, d, e, f are all positive and

$$a^2 + b^2 + c^2 = 16$$

$$d^2 + e^2 + f^2 = 49$$

$$ad + be + cf = 28.$$

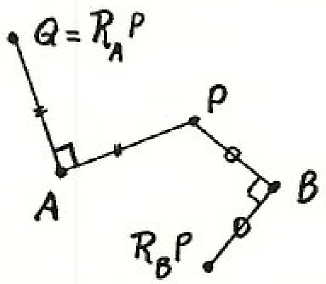
Determine the value of $\dfrac{a+b+c}{d+e+f}$.

Q.844 Every point on the circumference of a circle is coloured red, blue, or green. Show that however this is done it is possible to find three points A, B, C on the circle all of the same colour, such that $\triangle ABC$ is isosceles.

Q.845 The number 1991 is palindromic (it reads the same backwards as forwards) and its prime factors 11 and 181 are also palindromes. Find the smallest and the largest four digit palindromic numbers which factorise into two palindromic primes.

Q.846 Given five points U, V, W, X, Y (forming the vertices of a convex pentagon in the plane). Show how to construct with ruler and compasses a sixth point Z such that there can be found a hexagon $ABCDEF$ having U, V, W, X, Y, Z as mid-points of AB, BC, CD, DE, EF, FA respectively.

Q.847 A and B are two fixed points in the plane. If P is any point in the plane $\mathcal{R}_A P$ denotes the point Q obtained from P by a quarter turn anticlockwise rotation about A. (i.e. $P\hat{A}Q = 1$right angle, and $AP = AQ$). Similarly $\mathcal{R}_B P$ is the point to which P is moved by a quarter turn anticlockwise rotation about the point B.



If P_0 is a point in the plane, consider the sequence of points $P_0, P_1, P_2, \cdots, P_k, \cdots$ such that $P_1 = \mathcal{R}_A P_0$, $P_2 = \mathcal{R}_B P_1$, $P_3 = \mathcal{R}_A P_2, \cdots$. i.e. $P_{k+1} = \mathcal{R}_A P_k$ if k is even, and $P_{k+1} = \mathcal{R}_B P_k$ if k is odd. Find P_0 such that P_{1991} is the same point as P_0.

Q.848 In a certain quiz game, a contestant has the chance to win a prize by guessing which one of three boxes contains it. After he announces his guess, the presenter points to one of the remaining two boxes, informs him that the prize is **not** in the indicated box, and asks him if he wishes to stick to his original guess, or to try again.

What is his best strategy, and what is his probability of winning the prize?

Q.849 A committee comprising 14 women and 6 men is to be seated in 20 chairs around a circular table. In how many different ways can one choose the six chairs to be occupied by the men, if no two men are to be in adjacent seats?

Q.850 From the level top of a mountain one would have a perfect view of the surrounding countryside if it were not for the ruins of a square fortress, of side length 20 metres. A person standing d metres from the building finds that it obstructs $\frac{1}{6}$th of the panoramic view (i.e. the fortress subtends an angle of 60° at the point at which he is standing). Find the smallest and largest possible values of d.

Q.851 Observe that the number 1991 can be expressed as the sum of distinct divisors of 2000:-

$$1991 = 1000 + 500 + 400 + 50 + 40 + 1$$

$$\text{or} \qquad 1991 = 1000 + 500 + 250 + 125 + 100 + 16$$

(i) Find the smallest number N which cannot be expressed as the sum of distinct divisors of 2000.

(ii) Prove that some number less than N is expressible in at least 217 different ways as the sum of distinct divisors of 2000.