SOLUTIONS OF PROBLEMS 829–839

Q.829 (i) Let c be any integer. Show that the remainder when c^2 is divided by 4 cannot be either 2 or 3.

(ii) Let x be a positive integer, $A = 2x - 1$, $B = 5x - 1$, $C = 13x - 1$. Show that any two of A, B, C may be perfect squares, but that it is impossible that all three are squares.

ANS. (i) For use also in Q.834 we shall prove the somewhat stronger results:

(a) If x is odd then $x^2 = 8k + 1$ for some integer k.

(b) If x is even then $x^2 = 8k$ or $8k + 4$ for some integer k.

(a) x, being odd, differs by 1 from a multiple of 4:-

$x = 4m \pm 1$ for some integer m.

Then $x^2 = 8(2m^2 \pm m) + 1$.

(b) If x is twice an odd number then using (a), $x^2 = 2^2 \times (8k+1) = 8(4k)+4$ for some integer k.

Otherwise x is a multiple of 4 and x^2 is a multiple of 16.

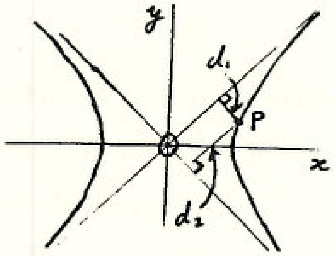
(ii) Obviously A is odd. Using (i) (a), if A is a square it is $8k + 1$ for some k, so $x = \frac{(8k+1)}{2} + 1 = 4k + 1$, an odd number. Hence B and C are even. Hence if all three are squares there are integers a (odd), b and c with $A = a^2$, $B = 4b^2$, and $C = 4c^2$.

Since $8A - 11B + 3C = 0$, after dividing by 4 we obtain $2a^2 = 11b^2 - 3c^2$.

The RHS is even only if b, c are both even, or else both odd. In either case, using (i) the RHS is then a multiple of 4. (e.g. $11(4k_1 + 1) - 3(4k_2 + 1) = 4(11k_1 - 3k_2 + 2)$). But the LHS is twice an odd number.

Hence it is impossible that all of A, B, C are squares. Taking $x = 1, x = 5$ and $x = 2$ shows that any two of A, B, C can be perfect squares.

Q.830 The curve with the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is called a hyperbola. The straight lines $\frac{x}{a} \pm \frac{y}{b} = 0$, through the origin are called asymptotes. If $a = b$ the asymptotes are at rightangles, and the hyperbola is called a rectangular hyperbola.



Let the perpendicular distances from a point P on a rectangular hyperbola to the asymptotes be d_1, and d_2. Show that $d_1 \times d_2 = 2a^2$. Deduce that by taking new axes along the asymptotes and adjusting the unit of length appropriately, any rectangular hyperbola can be given the Cartesian equation $XY = 1$.

ANS. If you attempted this question you probably noticed that $d_1 \times d_2 = a^2/2$, not $2a^2$. (e.g. Try $P = (a, 0), d_1 = d_2 = a/\sqrt{2}$). My apologies for the inaccuracy.

We use the theorem that the distance, d, from a point (x, y) to a line whose equation is in "perpendicular form" $(\cos\alpha)x + (\sin\alpha)y = p$, is given by $d = |(\cos\alpha)x_1 + (\sin\alpha)y_1 - p|$.

Then d_1, the distance to the asymptote $\frac{x}{\sqrt{2}} - \frac{y}{\sqrt{2}} = 0$ is $|\frac{x_1}{\sqrt{2}} - \frac{y_1}{\sqrt{2}}|$, and d_2, the distance to the asymptote $\frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}} = 0$ is $|\frac{x_1}{\sqrt{2}} + \frac{y_1}{\sqrt{2}}|$.

$$\therefore d_1 \times d_2 = |\frac{x_1^2 - y_1^2}{2}| = \frac{a^2}{2},$$

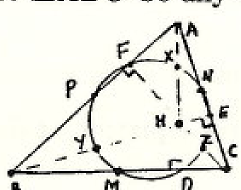
since $P(x_1, y_1)$ lies on the hyperbola $x^2 - y^2 = a^2$.

If we choose new axes OX, OY along the asymptotes with a new unit of length equal to k times the unit of length in the O_x, O_y co-ordinate system, the coordinates of P are now (X_1, Y_1) where $kX_1 = d_1$ and $kY_1 = d_2$.

$$\therefore k^2 X_1 Y_1 = d_1 d_2 = \frac{a^2}{2}.$$

If $k = \frac{a}{\sqrt{2}}$, this simplifies to $X_1 Y_1 = 1$. Hence the equation of the hyperbola in the OX, OY system is now $XY = 1$.

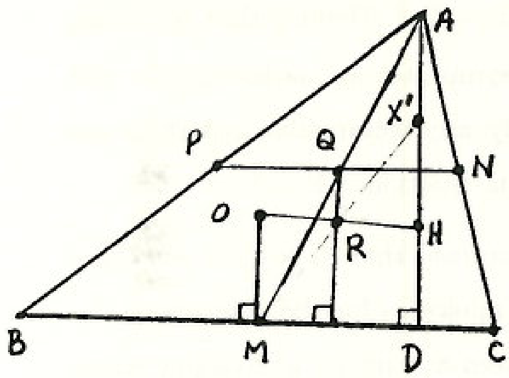
Q.831 Let $\triangle ABC$ be any triangle.



Let MN, P be the mid-points of the sides, D, E, F the feet of the altitudes, H the orthocentre and X, Y, Z the mid points of HA, HB, HC respectively.

Show that the points $M, N, P, D, E, F, X, Y, Z$ all lie on one circle. This is called the nine point circle of $\triangle ABC$.

ANS. It will be sufficient to prove that the circle through the mid-points of the sides, MNP, passes through D and X.



Let O be the circumcentre of ΔABC, so that $OM \perp BC$; let R be the mid-point of OH; and let the lines AM and PN intersect at Q. The diagonals of the parallelogram $MNAP$ intersect each other at Q. The perpendicular bisector of PN passes through Q and is parallel to OM and AD, so it bisects not only AM, but also both OH (at R) and MD.

Since similar reasoning shows that R is also on the perpendicular bisectors of PM and MN, it is the circumcentre of ΔMNP. Since R lies on the perpendicular bisector of MD, it is equidistant from M and D, whence D lies on the circle MNP.

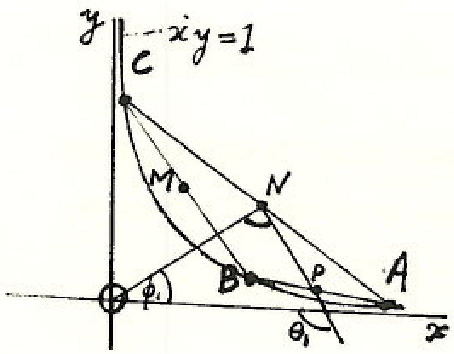
Now let X' be the point of intersection of the circle $MDNP$ with AD. We shall prove that $AX' = HX'$ so that X' coincides with X.

Since $\angle X'DM = 90°$, $X'M$ is a diameter of this circle, so R is the mid point of MX'. Since $\Delta HRX' \equiv \Delta ORM$, $HX' = OM$. (1)

Since Q, R are mid-points of MA and MX', $AX' = 2QR$. Also since ΔABC is similar to ΔMNP, with magnification factor equal to 2, the distance from BC to the circumcentre of ΔABC is twice the distance from NP to the circumcentre of ΔMNP; i.e. $OM = 2RQ = AX'$. Now using (1), $HX' = AX'$. Hence X' coincides with X, and the proof is complete.

Q.832 Let A, B, C be distinct points all lying on a rectangular hyperbola. Show that the centre of the hyperbola (the point of intersection of the asymptotes) lies on the nine point circle of ΔABC.

ANS. For simplicity we shall assume that A, B, C all lie on the same arm of the hyperbola. By Q.830 we can choose co-ordinates so that the equation of the hyperbola is $xy = 1$.



Let the three points be $A(a, \frac{1}{a})$, $B(b, \frac{1}{b})$, $C(c, \frac{1}{c})$ with $a > b > c > 0$. The mid-points of the sides are $M(\frac{b+c}{2}, \frac{\frac{1}{b}+\frac{1}{c}}{2})$, $N(\frac{a+c}{2}, \frac{\frac{1}{a}+\frac{1}{c}}{2}$829 and $P(\frac{a+b}{2}, \frac{\frac{1}{a}+\frac{1}{b}}{2})$. It is sufficient to show that $O(0,0)$, M, N, P are concyclic points, which can be achieved by showing that $\angle OMP = \angle ONP$.

$\angle ONP = \theta_1 - \phi_1$ where θ_1 is the angle made with O_x by NP produced, and ϕ is the angle made with O_x by ON.

$$\therefore \tan \angle ONP = \frac{\tan\theta_1 - \tan\phi_1}{1 + \tan\theta_1 \tan\phi_1}.$$

Now $\tan\theta_1$ =gradient of PN =gradient of BC (since $PN \parallel BC$) $= \frac{\frac{1}{c} - \frac{1}{b}}{c - b} = -\frac{1}{bc}$ and $\tan\phi_1$ = gradient of $ON = \frac{\frac{1}{a} + \frac{1}{c}}{a + c} = \frac{1}{ac}$.

$$\therefore \tan \angle ONP = \frac{-\frac{1}{bc} - \frac{1}{ac}}{1 - \frac{1}{bc} \cdot \frac{1}{ac}}.$$

$$\text{Simililarly} \tan \angle OMP = \frac{\text{gradient of } PM - \text{gradient of } OM}{1 + \text{ gradient of } PM \times \text{ gradient of } OM}$$
$$= \frac{-\frac{1}{ac} - \frac{1}{bc}}{1 - \frac{1}{ac} \cdot \frac{1}{bc}}$$

Since $\tan \angle ONP$, the angles are equal and the proof is complete.

Q.833 If x is any positive integer, $f(x)$ denotes the new integer obtained when the last digit of x (using the usual decimal representation) is transferred to the other end; e.g. $f(1356) = 6135$. Find the smallest integer such that $f(x) = 7 \times x$.

ANS. Let $y = 7x$. Since y has the same number of digits as x, the first digit of y must be 7,8, or 9.

If y begins with 7 we perform the following division, at each step appending the new digit just obtained in the quotient as the next digit in the dividend.

$$7\overline{)7\;{}^{0}1\;{}^{1}0\;{}^{3}1\cdots}$$

$$\nearrow\nearrow\nearrow$$

$$1\;\;0\;\;1\cdots$$

(See fig.1. The next digit to be obtained in the quotient is a 4. It will be placed after the $\cdots 01$ on both lines). The process can come to an end only when we obtain a 7 as the next digit in the quotient, with 0 to carry.

The complete calculation yields

$$7\overline{)7101449275362318840579}$$

$$1014492753623188405797$$

The same procedure applied to a number beginning with 8 or 9 halts (with the digit 8 or 9 respectively) after the same number of steps, yielding somewhat larger quotients. Hence the smallest integer x such that $f(x) = 7x$ is the quotient in the above calculation.

Q.834 Let $N(n)$ denote the number of different solutions in non- negative integers w, x, y, z of the equation $w^2 + x^2 + y^2 + z^2 = 3 \times 2^n$.

For example, $N(0) = 4$ since the only solutions of $w^2 + x^2 + y^2 + z^2 = 3$ are $(w, x, y, z) = (1,1,1,0)$ or $(1,1,0,1)$ or $(1,0,1,1)$ or $(0,1,1,1)$. Find $N(1991)$.

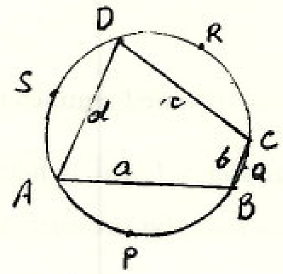
ANS. If $n \geq 3$ $\;w^2 + x^2 + y^2 + z^2 = 3 \times 2^n \equiv 0 \pmod 8$. Since $X^2 \equiv 1 \pmod 8$ if X is odd, and $X^2 \equiv 0$ or $4 \pmod 8$ if X is even (see Q.829), it is easy to check that it is impossible to have $w^2 + x^2 + y^2 + z^2 \equiv 0 \pmod 8$ if any of w, x, y, z are odd. Hence every solution of $w^2 + x^2 + y^2 + z^2 = 3 \times 2^n$ (with $n \geq 3$) is of the form $(2W)^2 + (2X)^2 + (2Y)^2 + (2Z)^2 = 3 \times 2^n$ where (W, X, Y, Z) is a solution of $W^2 + X^2 + Y^2 + Z^2 = 3 \times 2^{n-2}$ (and, conversely, for each such (W, X, Y, Z), $(2W, 2X, 2Y, 2Z)$ is a solution of $w^2 + x^2 + y^2 + z^2 = 3 \times 2^n$).

Hence $N(n) = N(n-2)$ if $n \geq 3$.

Thus $N(1991) = N(1989) = \cdots = N(3) = N(1)$.

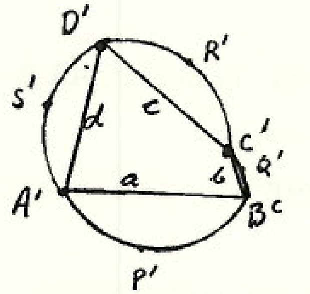
Since $w^2 + x^2 + y^2 + z^2 = 3 \times 2^1$ for (w, x, y, z) equal to any of the twelve arrangements of (2,1,1,0), $N(1) = 12 = N(1991)$.

Q.835 It is known that the region of maximum area having given perimeter p is the circular disc with radius $\dfrac{p}{2\pi}$.



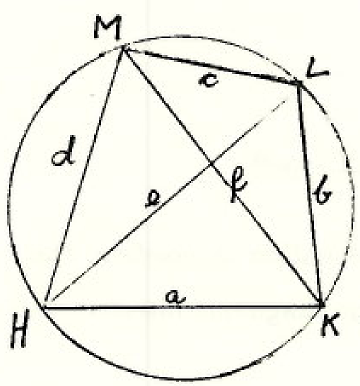
Assuming this or otherwise prove that of all quadrilaterals with sides of given lengths a, b, c, d that of maximum area is cyclic.

ANS. Construct on the sides of the second quadrilateral segments of circles congruent with the corresponding pieces in the first diagram. The figure $A'P'B'Q'C'R'D'S'A'$ has the same perimeter as the circle, hence its area must be smaller. When the areas of the congruent segments are subtracted, we are left with



$$\text{Area } A'B'C'D' < \text{ Area } ABCD \qquad \text{(QED)}.$$

Q.836 Let e, f be the lengths of the diagonals of a cyclic quadrilateral with sides of lengths a, b, c, d (see figure).



Show that (i) $e(ab + cd) = f(ad + bc)$

(Hint: See Q.818).

(ii) $e^2 = \dfrac{(ac + bd)(ad + bc)}{(ab + cd)}$

$f^2 = \dfrac{(ac + bd)(ab + cd)}{(ad + bc)}$

(You may assume Ptolemy's Theorem: $ef = ac + bd$).

ANS. (i) (It was shown in Q.818 that if a, b, c are the side lengths of a triangle of area A inscribed in a circle of radius R then $abc = 4AR$.)

Let R denote the radius of the circle $HKLM$. By the above result

$$eab + ecd = 4R(\text{Area } \Delta HKL) + 4R(\text{Area } HLM)$$

$$e(ab + ed) = 4R(\text{Area of } HKLM).$$

$$\text{Similarly } f(ad + bc) = 4R(\text{Area } \Delta HKM + \text{ Area } \Delta KLM)$$

$$= 4R(\text{Area of } HKLM).$$

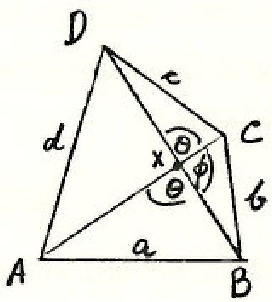
Hence the result.

(ii) Since from (i) $f = e\dfrac{(ab + cd)}{(ad + bc)}$

$$(ac+bd) = ef = e^2\frac{(ab+cd)}{ad+bc}$$

whence $e^2 = \dfrac{(ac+bd)(ad+bc)}{ab+cd}$. The expression for f^2 is obtained similarly.

Q.837 Let the sides in order around any quadrilateral have lengths a, b, c, d, where $a^2+c^2 = b^2+d^2$. Prove that the area of the quadrilateral is half the product of the lengths of the diagonals.



ANS. Let the diagonals intersect at X (fig. 1) and let $\angle AXB = \angle CXD = \theta$, $\angle BXC = \angle AXD = \phi$.

Since $\theta + \phi = 180°$, $\cos\phi = -\cos\theta$ (1)

Using the cosine rule

$a^2+c^2 = (AX^2+BX^2-2AX.BX\cos\theta) + (CX^2+DX^2-2CX.DX\cos\theta)$

and $b^2+d^2 = (BX^2+CX^2-2BX.CX\cos\phi)+(DX^2-AX^2-2DX.AX\cos\phi)$.

Since $a^2+c^2 = b^2+d^2$, we have using (1)

$$(AX.BX+CX.DX)\cos\theta = -(BX.CX+DX.AX)\cos\theta.$$

This is impossible if $\cos\theta \neq 0$, since one side of the equation is positive, the other negative. Hence we must have $\cos\theta = 0$, i.e. the diagonals intersect at right-angles.

$$\begin{aligned}\therefore\ \text{Area } ABCD &= \text{Area } \triangle ACD + \text{Area } \triangle ACB\\ &= \frac{1}{2}AC.DX + \frac{1}{2}AC.XB\\ &= \frac{1}{2}AC(DX+XB) = \frac{1}{2}AC.DB. \qquad \text{(QED)}.\end{aligned}$$

Q.838 Rods of lengths 60, 52, 39, and 25 units are joined together at their end points in any order to make a plane quadrilateral. Calculate the maximum possible value of its area. (Note that $60^2+25^2 = 50^2+39^2$).

ANS. If the sides of length 60 and 25 are opposite, by Q837 the area of the quadrilateral is $\frac{1}{2}ef$ where e and f are the lengths of the diagonals. The maximum area occurs when the quadrilateral is cyclic, by Q.835, and then using Q.836,

$$\text{Area} = \frac{1}{2}\sqrt{\frac{(ac+bd)(ad+bc)}{(ab+cd)}\frac{(ac+bd)(ab+cd)}{(ad+bc)}} = \frac{1}{2}(ac+bd)$$
$$= \frac{1}{2}(60 \times 25 + 52 \times 39). \qquad (1)$$

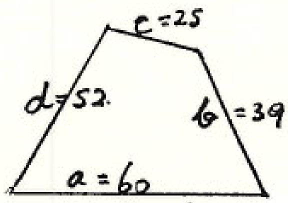


Fig 1

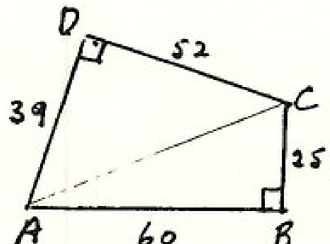


Fig 2.

If the sides of length 60 and 25 are adjacent, meeting at the vertex B (see fig.2) the maximum area is attained by making $\angle B = 90°$, since then by Pythagoras theorem and its converse $\angle D$ is also equal $90°$, and the quadrilateral is cyclic. The area of the quadrilateral is now area $\triangle ABC+$ area $\triangle ACD = \frac{1}{2}(60 \times 25 + 52 \times 39)$ (2)

Comparing (1) and (2) we see that a maximum area of 1764 sq units is achievable irrespective of the order in which the rods are joined.

Q.839 Prove that $\sqrt[3]{40+\sqrt{1573}} + \sqrt[3]{40-\sqrt{1573}}$ is **exactly** equal to 5.

ANS. Let $x = \sqrt[3]{40+\sqrt{1573}} + \sqrt[3]{40-\sqrt{1573}}.$

Since $(u+v)^3 = u^3+v^3+3uv(u+v)$ we have

$$x^3 = (40+\sqrt{1573}) + (40-\sqrt{1573}) + 3\sqrt[3]{(40+\sqrt{1573})(40-\sqrt{1573})}(x)$$
$$= 80 + 3\sqrt[3]{27}x = 80 + 9x$$

$x^3 - 9x - 80 = 0.$

It is evident that $x = 5$ is a root $(125 - 45 - 80 = 0)$ and since $x^3 - 9x - 80 = (x-5)(x^2+5x+16)$ there are no other real roots. Hence x must equal 5 exactly.

Correct solution: G. Cheong (Sydney Boys High School).