SOLUTIONS OF PROBLEMS 840 – 851

Q.840 (i) Let α, β be two distinct solutions of

$$x^3 - x^2 - x + c = 0$$

Simplify $\alpha^2\beta + \alpha\beta^2 - \alpha\beta$.

(ii) Let α, β be two distinct solutions of

$$x^4 + x^3 + kx^2 - x - 1 = 0.$$

Simplify $\alpha^3\beta^2 + \alpha^2\beta^3 + \alpha^2\beta^2 + \alpha\beta + \alpha + \beta$.

ANS. (i) Let the third root of the equation be γ. Then $\alpha + \beta + \gamma = -(-1)$ and $\alpha\beta\gamma = -c$.

Eliminating γ, we obtain $\alpha\beta(1 - \alpha - \beta) = -c$.

Removing brackets and changing the sign gives $\alpha^2\beta + \alpha\beta^2 - \alpha\beta = c$.

(ii) Let the remaining two roots be γ, δ.

$$\text{Then } \alpha + \beta + \gamma + \delta = -1 \quad (1)$$

$$\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = 1 \quad (2)$$

$$\alpha\beta\gamma\delta = -1 \quad (3)$$

Multiply (2) throughout by $\alpha\beta$.

$$\alpha^2\beta^2(\gamma + \delta) + (\alpha + \beta)\alpha\beta\gamma\delta = \alpha\beta$$

$$\alpha^2\beta^2(-1 - \alpha - \beta) + (\alpha + \beta) \times (-1) = \alpha\beta$$

$$\therefore \alpha^3\beta^2 + \alpha^2\beta^3 + \alpha^2\beta^2 + \alpha\beta + \alpha + \beta = 0.$$

Correct solution: D. Thadani (Sydney Boys High School).

Q.841 None of a, b, c is zero, and α, β are the roots of $ax^2 + bx - c = 0$.

If $2\alpha, 2\beta$ are the roots of $a^2x^2 + b^2x + c^2 = 0$ find the roots of $a^3x^2 + b^3x - c^3 = 0$.

ANS. We have $\alpha+\beta=-\frac{b}{a}$, $\alpha\beta=-\frac{c}{a}$, $2\alpha+2\beta=-\frac{b^2}{a^2}$ and $(2\alpha)(2\beta)=-\frac{c^2}{a^2}$. From the first and third of these $\frac{b}{a}=2$, and from the second and fourth $\frac{c}{a}=4$.

The roots of $a^3x^2+b^3x-c^3\equiv a^3(x^2+8x-64)=0$ are then $x=\dfrac{-8\pm\sqrt{8^2+4\times 64}}{2}=4(-1\pm\sqrt{5})$.

(Since α,β are the roots of $a(x^2+2x-4)=0$; viz. $\alpha,\beta=-1\pm\sqrt{5}$, the required roots are $4\alpha,4\beta$).

Correct solution: B. Morris (Sydney Grammar School).

Q.842 Prove that if for some non-negative number c

$$\frac{a_1}{c+1}+\frac{a_2}{c+2}+\frac{a_3}{c+3}+\cdots+\frac{a_n}{c+n}=0$$

then the equation

$$a_1+a_2x+a_3x^2+\cdots+a_nx^{n-1}=0$$

has a solution x lying between 0 and 1.

ANS. Let $P(x)=a_1+a_2x+a_3x^2+\cdots+a_nx^{n-1}$ (the L.H.S. of the equation).
Note that $x^cP(x)=\dfrac{d}{dx}\left(\dfrac{a_1x^{c+1}}{c+1}+\dfrac{a_2x^{c+2}}{c+2}+\cdots+\dfrac{a_nx^{c+n}}{c+n}\right)$ so that
$\displaystyle\int_0^1 x^cP(x)dx=\frac{a_1}{c+1}+\frac{a_2}{c+2}+\cdots+\frac{a_n}{c+n}=0.$
Hence the function $x^cP(x)$ cannot be positive for all x in (0,1), (since then $\int_0^1 x^cP(x)dx$ would also be positive); nor can it be negative for all x in (0,1). Since it is a continuous function having both positive and negative values in (0,1), it must attain the value 0 at some point, α, in (0,1). Since $\alpha^c\neq 0$ at this point, we must have $P(\alpha)=0$.

[Comment: (0,1) denotes the interval of all real numbers lying between 0 and 1.]

Q.843 The numbers a,b,c,d,e,f are all positive and

$$a^2+b^2+c^2=16$$

$$d^2+e^2+f^2=49$$

$$ad+bc+cf=28.$$

Determine the value of $\dfrac{a+b+c}{d+e+f}$.

ANS.

$$\begin{aligned}
&(7a-4d)^2+(7b-4a)^2+(7c-4f)^2\\
&=49(a^2+b^2+c^2)-56(ad+bc+cf)+16(d^2+c^2+f^2)\\
&=49\times 16-56\times 28+16\times 49=0.
\end{aligned}$$

Since squares of real numbers are never negative we must have

$$\begin{aligned}
0 &= 7a-4d=7b-4c=7c-4f\\
\therefore a+b+c &= \frac{4}{7}d+\frac{4}{7}c+\frac{4}{7}f\\
\text{so that } \frac{a+b+c}{d+c+f} &= \frac{4}{7}.
\end{aligned}$$

Q.844 Every point on the circumference of a circle is coloured red, blue, or green. Show that however this is done it is possible to find three points A, B, C on the circle all of the same colour, such that $\triangle ABC$ is isosceles.

ANS. Consider any set of thirteen points equally spaced around the circle; $P_1, P_2, \cdots, P_{13}$.

There must be at least 5 of these points which have the same colour. It is not possible to choose 5 of the vertices of a regular 13-gon without some subset of three of them forming the vertices of an isosceles triangle. The result is now clear.

Correct solution: Jonathon Kong (Sydney Grammar School) showed instead that if the vertices of a regular 11-gon are coloured with 3 colours, an isosceles triangle cannot be avoided.

Q.845 The number 1991 is palindromic (it reads the same backwards as forwards) and its prime factors 11 and 181 are also palindromes. Find the smallest and the largest four digit palindromic numbers which factorise into two palindromic primes.

ANS. Every 4 digit palindrome, $abba$, is divisible by 11, since $1001a + 110b = 11(91a + 10b)$.

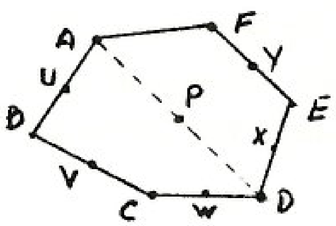
Since there is no other two digit palindromic prime except 11, the smallest 4-digit palindromic number which factorises into two palindromic primes must be $11 \times 101 = 1111$.

If a three digit number $91a + 10b$ is palindromic, its decimal representation is $aca = 101a + 10c$ whence we deduce that $a + c = b$. If aca is also a prime greater then 101, a must be odd, and $c > 0$ (since a is a factor of $a0a$.) Also $c = b - a < 10 - a$.

Thus $a \neq 9$ since there is no digit c such that $0 < c < 10 - 9$. If $a = 7$, $0 < c < 3$. Since 727 proves to be a prime number, the large palindrome we are looking for is $11 \times 727 = 7997$.

Q.846 Given five points U, V, W, X, Y (forming the vertices of a convex pentagon in the plane). Show how to construct with ruler and compasses a sixth point Z such that there can be found a hexagon $ABCDEF$ having U, V, W, X, Y, Z as mid-points of AB, BC, CD, DE, EF, FA respectively.

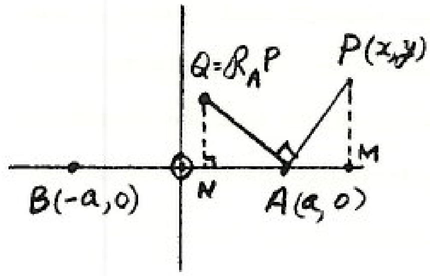
ANS. Analysis: Let $ABCDEF$ be any hexagon having U, V, W, X, Y as mid points of AB, BC, CD, DE, and EF respectively.



Let P be the mid point of AD. Then $UVWP$ is a parallelogram (each of UV and PW is $\| AC$ and of length $\frac{1}{2}AC$).
Similarly, if Z is the mid point of AF then $PXYZ$ is also a parallelogram.

Hence there is only one possible point Z and it can be constructed as follows: construct a line through $U \| VW$, and a line through $W \| VU$. Let P be their point of intersection. Construct a line through $P \| XY$, and a line through $Y \| XP$. Their point of intersection is the required point Z.

Q.847 A and B are two fixed points in the plane. If P is any point in the plane $\mathcal{R}_A P$ denotes the point Q obtained from P by a quarter turn anticlockwise rotation about A. (i.e. $P\hat{A}Q =$ 1right angle, and $AP = AQ$). Similarly $\mathcal{R}_B P$ is the point to which P is moved by a quarter turn anticlockwise rotation about the point B.



If P_0 is a point in the plane, consider the sequence of points $P_0, P_1, P_2, \cdots, P_k, \cdots$ such that $P_1 = \mathcal{R}_A P_0$, $P_2 = \mathcal{R}_B P_1$, $P_3 = \mathcal{R}_A P_2, \cdots$. i.e. $P_{k+1} = \mathcal{R}_A P_k$ if k is even, and $P_{k+1} = \mathcal{R}_B P_k$ if k is odd. Find P_0 such that P_{1991} is the same point as P_0.

ANS. Choose axes with 0 at the mid point of AB, and the x-axis containing A and B. Let A have co-ordinates $(a, 0)$ so that the co-ordinates of B are $(-a, 0)$. Let P have co-ordinates (x, y). Then if Q is the point $\mathcal{R}_A P, \triangle APM \equiv \triangle QAN$ (see figure) whence $NQ = AM = x - a$, and $NA = MP = y$ so that the co-ordinates of $\mathcal{R}_A P$ are $(a - y, x - a)$.

Similarly the co-ordinates of $\mathcal{R}_B P$ are $(-a - y, x + a)$. Now let the co-ordinates of P_k be (x_k, y_k). Then $P_1 = \mathcal{R}_A P_0$ so $(x_1, y_1) = (a - y_0, x_0 - a)$;

$P_2 = \mathcal{R}_B P_1$ so $(x_2 y_2) = (-a - y_1, x_1 + a) = (-x_0, 2a - y_0)$

$P_3 = \mathcal{R}_A P_2$ so $(x_3, y_3) = (a - y_2, x_2 - a) = (-a + y_0, -x_0 - a)$

$P_4 = \mathcal{R}_B P_3$ so $(x_4, y_4) = (-a - y_3, x_3 + a) = (x_0, y_0)$.

Thus $P_4 = P_0$. Repeating these steps shows that $P_0 = P_4 = P_8 = P_{4n} = P_{1988}$, and that $P_{1991} = P_3$.

$$\therefore (x_{1991}, y_{1991}) = (-a + y_0, -x_0 - a).$$

Thus P_{1991} is the same point as P_0 iff $(x_0, y_0) = (-a + y_0, -x_0 - a)$

$$\therefore x_0 = -a + y_0 \text{ and } y_0 = -x_0 - a$$

Hence P_0 must coincide with the point $B(-a, 0)$.

Q.848 In a certain quiz game, a contestant has the chance to win a prize by guessing which one of three boxes contains it. After he announces his guess, the presenter points to one of the remaining two boxes, informs him that the prize is **not** in the indicated box, and asks him if he wishes to stick to his original guess, or to try again.

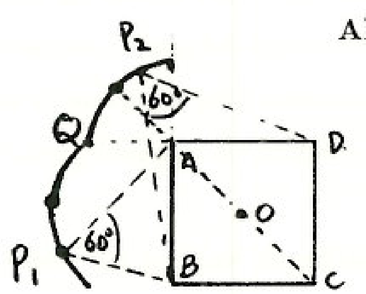
What is his best strategy, and what is his probability of winning the prize?

ANS. There is probability $\frac{1}{3}$ that the initial guess was correct; and $\frac{2}{3}$ that it was incorrect. After the presenter's additional information, there is probability 0 that the prize is in the box he indicated, so the probability that it is in the remaining box is $\frac{2}{3}$. Hence the contestant's best strategy is to change his guess to the box not indicated by the presenter, so boosting his winning chance from $\frac{1}{3}$ to $\frac{2}{3}$.

Q.849 A committee comprising 14 women and 6 men is to be seated in 20 chairs around a circular table. In how many different ways can one choose the six chairs to be occupied by the men, if no two men are to be in adjacent seats?

ANS. A slightly simpler problem would be to seat 6 men and 14 women in 20 chairs in a straight row if the chair to the right of each man is to hold a woman. This reduces to the number of different arrangements of $6(MW)$ "pairs" and 8 W's; viz ${}^{14}C_6$. This answer is too small for the given question, since none of the above arrangements have a man in the right hand end chair. If the 20 chairs are in a circle, the number of additional arrangements with a man in that chair, (and hence a woman in the next chair to the right) is the number of ways of placing 5 MW pairs and 8 women in the remaining eighteen chairs. viz ${}^{13}C_5$.
Hence the required answer is ${}^{14}C_6 + {}^{13}C_5 = 4290$.

Q.850 From the level top of a mountain one would have a perfect view of the surrounding countryside if it were not for the ruins of a square fortress, of side length 20 metres. A person standing d metres from the building finds that it obstructs $\frac{1}{6}$th of the panoramic view (i.e. the fortress subtends an angle of 60° at the point at which he is standing). Find the smallest and largest possible values of d.



ANS. In the figure, P_1 and P_2 are two points at which the fortress obstructs 60° of the view. The locus of P_1 is an arc of a circle at the circumference of which the side AB subtends an angle of 60°. The distance from P_1 to the fortress is the perpendicular distance to AB, which decreases from the mid point of the arc (where ΔPAB is equilateral, and $d = 10 \times \tan 60° = 10 \times \sqrt{3}$ metres) to the point Q on DA produced (where $d = 20 \tan 30° = \frac{20\sqrt{3}}{3}$ metres).

The distance from P_2 to the fortress is the length AP_2 which decreases from AQ to its value at the mid point of the second arc (where ΔP_2BD is equilateral and $d = OP_2 - OA = 10\sqrt{2}\tan 60° - 10\sqrt{2} = 10\sqrt{2} - (\sqrt{3} - 1)$ metres). Thus the smallest and longest values of d are $10\sqrt{2}(\sqrt{3} - 1)$ metres and $10\sqrt{3}$ metres respectively.

Correct solutions: J. Kong (Sydney Grammar School), and another without identification.

Q.851 Observe that the number 1991 can be expressed as the sum of distinct divisors of 2000:-

$$1991 = 1000 + 500 + 400 + 50 + 40 + 1$$

$$\text{or} \qquad 1991 = 1000 + 500 + 250 + 125 + 100 + 16$$

(i) Find the smallest number N which cannot be expressed as the sum of distinct divisors of 2000.

(ii) Prove that some number less than N is expressible in at least 217 different ways as the sum of distinct divisors of 2000.

ANS. (i) Note that $2000 = 2^4 \times 5^3$. When $(1 + 2 + 2^2 + 2^3 + 2^4)(1 + 5 + 5^2 + 5^3)$ is expanded every one of the twenty distinct factors of 2000 occurs exactly once in the resulting expression. Hence no number larger than $31 \times 156 = 4836$ can be expressed as the sum of distinct divisors of 2000. Hence we will have shown that $N = 4837$ if we show that every number up to 4836 is expressible in the stated form.

We shall prove by mathematical induction the more general assertion that for any positive integer k, every non negative whole number up to $31 \times \left(\frac{5^{k+1} - 1}{4}\right)$ is the sum of distinct divisors of $2^4 \times 5^k$. Denote this assertion by $P(k)$. $P(0)$ is clearly true, since it is a familiar observation that every number up to 31 can be expressed in binary notation as $a_4 \times 2^4 + a_3 \times 2^3 + a_2 \times 2^2 + a_1 \times 2 + a_0$ with every $a_t = 0$ or 1.

Assuming that $P(m)$ is true for some m let x be any non-negative integer up to $31 \times \left(\frac{5^{m+2}-1}{4}\right)$. The divisors of $2^4 \times 5^{m+1}$ include all the divisors of $2^4 \times 5^m$ and in addition the numbers in the set $S = \{2^j \times 5^{m+1} : j = 0, 1, 2, 3, \text{ or } 4\}$. For any whole number u up to 31, a subset of S can be found which adds up to $u \times 5^{m+1}$. Now consider any positive integer $x \leq 31 \times \left(\frac{5^{m+2}-1}{4}\right)$. Choose u to be the largest integer not exceeding $\frac{x}{5^{m+1}}$, or else 31 (whichever is smaller) and define $y = x - u \times 5^{m+1}$. Then $y \leq 31 \times \left(\frac{5^{m+2}-1}{4}\right) - 31 \times 5^{m+1} \leq 31 \times \left(\frac{5^{m+1}-1}{4}\right)$. Thus $x = y + u \times 5^{m+1}$ where $u \times 5^{m+1}$ is expressible as the sum of distinct elements of S, and y (by the induction assumption) is expressible as the sum of distinct divisors of $2^4 \times 5^m$.

Hence x is expressible as the sum of distinct divisors of $2^4 \times 5^{m+1}$.

This completes the induction step, and establishes $P(k)$ for all $k \in \mathbb{N}$.

(ii) There are twenty different divisors of 2000. In selecting a subset of these divisors, for each one we have two choices:- select it, or reject it. Hence there are 2^{20} different subsets, or $2^{20} - 1$ if we omit the empty subset. Hence there are $2^{20} - 1$ different expressions of sums of distinct divisors of 2000, and since the sum is in every case one of the numbers $1, 2, \cdots, 4836$ the **average** number of expressions yielding a prescribed answer is $\frac{2^{20}-1}{4836} > 217$. The result follows.

PUZZLE SOLUTION

C. Kenrick (Wolllongong H.S.) has sent his solution of the puzzle WHO'S SITTING WHERE? (Vol 27, No 2 p5).

For reasons of space, publication is held over until the next issue of parabola.