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# POLYNOMIAL INTERPOLATION

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## INTRODUCTION

Interpolation is the process of putting a curve of some sort through a series of points. For example, suppose we took temperature measurements every hour on the hour but then realised that we needed to know the temperature at 2.35pm. What would we do? One method commonly used is linear interpolation which involves drawing a straight line between the two points nearest to 2.35pm. Thus, if the temperature at 2pm was 20° and that at 3pm was 24° our estimate of the temperature at 2.35pm would be $20 + (24 - 20) \times 35/60) = 22\ 1/3°$. In this calculation we have used only the information contained in the measurements taken at 2pm and 3pm. One might suspect that a more accurate answer might be obtained by somehow incorporating the information contained in the measurements made at other times as well, for example 1pm and 4pm. How are we to do this? Polynomial interpolation is one such method which we shall describe in this article.

## THE PROBLEM

Suppose we have measured some function $f$ at the $(n+1)$ different points $x_0, x_1, \cdots x_n$. These points are often called **nodes**. That is to say, we have the $(n+1)$ pairs $(x_0, f(x_0))$, $(x_1, f(x_1)), \cdots, (x_n, f(x_n))$ which we may plot on graph paper if we wish. We now want to draw a curve which passes through all these points. Expressed mathematically, we are seeking a function $g$ such that $g(x_i) = f(x_i)$ for $i = 0, 1, 2, \cdots n$. For $x \neq x_i$ we then use $g(x)$ to estimate the value of $f(x)$. Clearly, there are an infinite number of such functions. For example, if $g_1$ is one such then $g_2$ defined by

$$g_2(x) = g_1(x) + a(x - x_0)(x - x_1) \cdots (x - x_n)$$

has the property that $g_2(x_i) = g_1(x_i) = f(x_i)$ for $i = 0, 1, 2, \cdots, n$. Here, $a$ is any constant. We could also replace $a$ by any continuous function, $\sin(x^2 + e^x)$ for example. So we have to be more specific and say precisely what type of function $g$ we want. Polynomial interpolation takes $g$ to be the polynomial of lowest possible degree which passes exactly through

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all the points. Since we have $(n+1)$ pieces of information contained in our measurements we would expect that our polynomial would have to contain $(n+1)$ coefficients and so be of the form $g(x) = a_0 + a_1 x + \cdots + a_n x^n$.

Thus, for two points $(n = 1)$ a straight line would do, for three points $(n = 2)$ we would need a quadratic, for four points $(n = 3)$ we would need a cubic and so on. So, in general, $g$ would be expected to be of degree $n$. However we should note that three points just might happen to lie on a straight line, four just might happen to lie on a parabola and so on. Hence, we would now expect that $g$ would be of degree at most $n$.

There are three problems to be addressed. First, how do we know that we can always construct our interpolating polynomial $g$? Second, how **do** we construct $g$? Third, how do we know that there are not other polynomials of degree at most $n$ which will also do the trick, in other words: is $g$ unique?

## THE LAGRANGE POLYNOMIAL

We will now construct the required interpolating polynomial. For each of the nodes $x_k$ we define the elementary Lagrange polynomial $L_k(x)$ by

$$L_k(x) = \prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{(x - x_i)}{(x_k - x_i)} \tag{1}$$

where $\prod_{i=0}^{n} b_i$ means the product $b_0 b_1 b_2 \cdots b_n$. Observe that in (1) we miss out the term $i = k$ in the factors on the top and bottom line. Thus $L_k(x)$ is a polynomial whose degree is exactly $n$. Further, $L_k(x_k) = 1$ since the top and bottom lines are then identical and at any of the other nodes $x_j$ for $j \neq k$, $L_k(x_j) = 0$ since there will then be a factor $(x_j - x_j)$ on the top line. In symbols

$$L_k(x_j) = \begin{cases} 0 & \text{for } j \neq k \\ 1 & \text{for } j = k. \end{cases}$$

We construct $L_k(x)$ for each of the nodes $x_k$ for $k = 0, 1, 2, \cdots n$. Each is a polynomial of degree $n$. We now form the function

$$\begin{aligned} p_n(x) &= \sum_{k=0}^{n} f(x_k) L_k(x) \\ &= f(x_0)L_0(x) + f(x_1)L_1(x) + \cdots + f(x_n)L_n(x) \end{aligned} \tag{2}$$

which is the sum of $(n+1)$ polynomials each of degree $n$. Thus $p_n(x)$ is a polynomial of degree at most $n$. We say 'at most $n$' because, if the values of the $f(x_i)$ are just right, the coefficients of $x^n$ might add up to zero. For example $2(x^2+x+1)+(x^2-6x+4)-3(x^2+10x+1)$ is a polynomial of degree 1 not 2. Further,

$$\begin{aligned} p_n(x_0) &= f(x_0)L_0(x_0) + f(x_1)L_1(x_0) + \cdots f(x_n)L_n(x_0) \\ &= 1f(x_0) + 0f(x_1) + \cdots + 0f(x_n) \\ &= f(x_0) \end{aligned}$$

since $L_i(x_0) = 0$ for $i \neq 0$. Similarly $p_n(x_i) = f(x_i)$ for $i = 0, 1, 2, \cdots n$.

Thus $p_n(x)$ is a polynomial of degree at most $n$ which passes exactly through our $(n+1)$ data points. It is sometimes called the Lagrange form of the interpolating polynomial.

We have thus proved that the required polynomial exists by demonstrating explicitly how to construct it. This sort of argument is sometimes called a **constructive proof** in mathematics.

There is still a problem however – how do we know that there are not two or more such polynomials; the others being constructed by some other techniques' before the? If there were, which one would we use? This problem is readily resolved. For, suppose $g_1(x)$ and $g_2(x)$ were two polynomials of degree at most $n$ which both passed exactly through the points $(x_i, f(x_i))$ for $i = 0, 1, 2, \cdots n$. Then $g_3$ defined by

$$g_3(x) = g_1(x) - g_2(x)$$

is also a polynomial of degree at most $n$ which has the value 0 at the $(n+1)$ nodes $x_i$. Since a non-trivial polynomial of degree $n$ can have at most $n$ distinct zeros, this is impossible. Thus $g_3(x)$ must take the value 0 for all $x$ which means that $g_1(x) = g_2(x)$ for all $x$. We have thus shown that $g$ is unique and so is given by $p_n$. This is called a **uniqueness proof** in mathematics.

## AN EXAMPLE

Find the polynomial of lowest possible degree passing through the points $(0,1)$, $(1,1)$, $(2,2)$ and $(4,5)$. Here $x_0 = 0, x_1 = 1, x_2 = 2$ and $x_3 = 4$. We first construct the elementary

Lagrange polynomials using (1)

$$L_0(x) = \frac{(x-1)(x-2)(x-4)}{(0-1)(0-2)(0-4)}$$
$$L_1(x) = \frac{(x-0)(x-2)(x-4)}{(1-0)(1-2)(1-4)}$$
$$L_2(x) = \frac{(x-0)(x-1)(x-4)}{(2-0)(2-1)(2-4)}$$
$$L_3(x) = \frac{(x-0)(x-1)(x-2)}{(4-0)(4-1)(4-2)}$$

Thus, using (2),

$$\begin{aligned} p_3(x) &= 1L_0(x) + 1L_1(x) + 2L_2(x) + 5L_3(x) \\ &= \cdots \text{ (much algebra)} \\ &= \frac{1}{12}(-x^3 + 9x^2 - 8x + 12). \end{aligned}$$

**DISCUSSION**

We have shown how to construct our interpolating polynomial. The main problem with the Lagrange construction is that if we add one or more new points to the data set, the whole process has to be done again from the beginning since all the elementary Lagrange polynomials will be changed. There is another way of constructing $p_n(x)$, the Newton form, which partially circumvents this problem, but that's another story.

There is a more fundamental problem however. That is that a polynomial of degree $n$ may have up to $(n-1)$ maxima or minima. Loosely speaking, it may have many oscillations. Although our $p_n(x)$ is constrained to pass exactly through the data points, it may exhibit large oscillations elsewhere which are unlikely to be representative of the actual, but unknown, function which we have measured only at the nodes. The problem here is that we are requiring one polynomial of high degree to represent the data. In some circumstances this is unwise and it is preferable to use a succession of different low order polynomials. For example we could just draw straight lines between the data points. This is just linear interpolation between the nodes as discussed earlier. A better choice, widely used in computer graphics, is to use cubic polynomials leading to what are called "cubic splines" which I hope to discuss in a later article.