SOLUTIONS OF PROBLEMS 852 – 860

Q.852 If $a_1, a_2, \cdots a_n$ are positive real numbers and $a_1 + a_2 + \cdots + a_n = 1$ prove that

$$\sum_{k=1}^{n}\left(a_k + \frac{1}{a_k}\right)^2 \geq \frac{(n^2+1)^2}{n}.$$

ANSWER. The problem has been corrected here. It appeared with the inequality in the wrong direction.

Let $f(x) = \left(x + \frac{1}{x}\right)^2 + \left(c - x + \frac{1}{c-x}\right)^2 = 4 + x^2 + \frac{1}{x^2} + (c-x)^2 + \frac{1}{(c+x)^2}$. Then $f''(x) = 2 + \frac{6}{x^4} + 2 + \frac{6}{(c-x)^4} > 0$ for all x. Hence the graph of $f(x)$ is concave upwards in $0 < x < c$. Thus $f(x)$ can have at most one stationary point in this interval, and if there is one it must be a minimum. Since $f'(x) = 2x - \frac{2}{x^3} - 2(c-x) + \frac{2}{(c-x)^3}$, $f'(\frac{c}{2}) = 0$. Hence $f(x)$ has a minimum value at $\frac{c}{2}$.

Now consider $\sum_{k=1}^{n}\left(a_k + \frac{1}{a_k}\right)^2$ with $\sum_{k=1}^{n} a_k = 1$, all $a_k > 0$.

If any two of the numbers a_k are different, say $a_i < a_j$, set $c = a_i + a_j$ and observe from the above that a smaller value of the sum is obtained if each of a_i, a_j is replaced by $\frac{c}{2}$.

Hence the minimum value of the sum can be obtained only when all the numbers a_k are equal, in fact all equal to $\frac{1}{n}$.

$$\therefore \sum_{k=1}^{n}\left(a_k + \frac{1}{a_k}\right)^2 \geq n\left(\frac{1}{n} + n\right)^2 = \frac{(n^2+1)^2}{n}.$$

Q.853 a) Show that for every positive integer n

$$2\left(\sqrt{n+1} - \sqrt{n}\right) < \frac{1}{\sqrt{n}} < 2\left(\sqrt{n} - \sqrt{n-1}\right)$$

b) Find the largest integer less than

$$\sum_{k=1}^{10000} \frac{1}{\sqrt{k}}.$$

ANSWER a)

$$\begin{aligned}\sqrt{n+1} - \sqrt{n} &= \frac{\left(\sqrt{n+1} - \sqrt{n}\right)\left(\sqrt{n+1} + \sqrt{n}\right)}{\left(\sqrt{n+1} + \sqrt{n}\right)} \\ &= \frac{1}{\sqrt{n+1} + \sqrt{n}} < \frac{1}{\sqrt{n} + \sqrt{n}} = \frac{1}{2\sqrt{n}}\end{aligned}$$

Similarly $2\left(\sqrt{n}-\sqrt{n-1}\right)\frac{\left(\sqrt{n}+\sqrt{n-1}\right)}{\left(\sqrt{n}+\sqrt{n-1}\right)} = \frac{2}{\sqrt{n}+\sqrt{n-1}} > \frac{2}{\sqrt{n}+\sqrt{n}} = \frac{1}{\sqrt{n}}$.

b)

$$1+\sum_{k=2}^{10000} 2\left(\sqrt{k+1}-\sqrt{k}\right) < \sum_{k=1}^{10000}\frac{1}{\sqrt{k}} < 1+\sum_{k=2}^{100000} 2\left(\sqrt{k}-\sqrt{k-1}\right)$$

$$1-2\sqrt{2}+2\sqrt{10001} < \sum_{k=1}^{10000}\frac{1}{\sqrt{k}} < 1-2\sqrt{1}+2\sqrt{10000}$$

$$\therefore 198 = 1-3+2\times 100 < \sum_{k=1}^{10000}\frac{1}{\sqrt{k}} < 199.$$

Therefore the required answer is 198.

Q.854 The number 4 can be expressed as the sum of one or more positive integers in 8 ways: 4, 3+1, 2+2, 2+1+1, 1+3, 1+2+1, 1+1+2, 1+1+1+1. Note that the order of the summands is regarded as significant in this count; e.g. $2+1+1$ and $1+2+1$ and $1+1+2$ are all counted. Find a formula for the number of different ways in which an arbitrary positive integer n can be expressed as a sum of positive integers.

ANSWER. Rayman Yan (Randwick Boys Technology High School) gives two solutions:-

1) Let N_n denote the number of ways to express a positive integer n in the form

$$n = a_1 + a_2 + \cdots + a_k \text{ where } k \geq 1, \text{ and } a_1, \cdots, a_k \text{ are positive integers.} \qquad (1)$$

Of the N_n ways, the number with $a_1 = m$ $(= 1, 2, \cdots n-1)$ is N_{n-m} (the number of ways of expressing $n-m$ in the form $a_2 + \cdots + a_k$). Obviously $a_1 = n$ gives one extra solution.

$$\therefore N_n = N_{n-1} + N_{n-2} + \cdots + N_1 + 1. \qquad (2)$$

Replacing n by $n-1$

$$N_{n-1} = N_{n-2} + N_{n-3} + \cdots + 1. \qquad (3)$$

Substituting (3) into (2) yields $N_n = 2N_{n-1}$ for all n.

Since $N_1 = 1$, we now have

$$N_n = \frac{N_n}{N_{n-1}} \times \frac{N_{n-1}}{N_{n-2}} \times \cdots \times \frac{N_2}{N_1} \times N_1$$
$$= 2^{n-1}.$$

2) For each of the N_n different solutions of (1)

$$1 + a_1 + a_2 + \cdots + a_k = n + 1$$

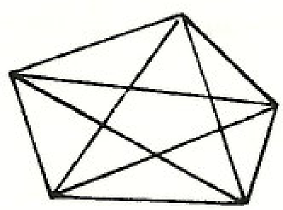
and

$$(a_1 + 1) + a_2 + \cdots + a_k = n + 1$$

are all different ways of expressing $(n+1)$ in the required form.

i.e. of the N_{n+1} expressions of the required form which sum to $n+1$, there are N_n having the first term equal to 1, and another N_n having the first term greater than 1. Hence $N_{n+1} = 2N_n$ for all n, and it follows as in the first method that $N_n = 2^{n-1}$.

Q.855 The diagram shows a convex pentagon with all diagonals drawn.



They intersect in 5 points, which divide the diagonals into 15 line segments. The diagonals partition the interior of the pentagon into 11 regions. Given a convex n-gon with all diagonals drawn, no three of which are concurrent, find

(i) how many points of intersection are there;

(ii) into how many segments are the diagonals divided;

(iii) the number of regions into which the n-gon is partitioned by the diagonals.

ANSWER (i) For each point of intersection, the two intersecting diagonals have as their end points a subset of 4 of the n vertices. Conversely for any selection of 4 vertices A, B, C, D in order around the perimeter, only the diagonals AC and BD intersect. Hence the number of intersections is ${}^nC_4 = \dfrac{n(n-1)(n-2)(n-3)}{4!}$, the number of different ways to choose subsets of 4 vertices.

(ii) First we count the number of diagonals. From each of the n vertices there are $(n-3)$ diagonals, which connect it to all the vertices except itself and its two neighbours. Hence there are $\frac{n \times (n-3)}{2}$ diagonals altogether. (The denominator is necessary because each diagonal has 2 ends). We now consider the effect on the number of segments of adding the ${}^nC_4$ points of intersection. Each of these increases the number of segments by one on both the diagonals that intersect. Hence the final number of segments must be

$$\begin{aligned}\frac{n(n-3)}{2} + 2 \times {}^n C_4 &= \frac{n(n-3)}{12}(6 + (n-1)(n-2)) \\ &= \frac{1}{12}n(n-3)(n^2 - 3n + 8)\end{aligned}$$

(iii) For a connected "map" in the plane having V vertices and E edges, the number of regions R into which the plane is divided is given by Euler's formula $V - E + R = 2$. (The unbounded "exterior" portion of the plane is included in counting R). Hence the number of regions **inside** the n-gon is given by $1 + E - V$ where $V = n + {}^n C_4$ and $E = n + \frac{1}{12}n(n-3)(n^2 - 3n + 8)$. (In the count of V we have added the n vertices of the n-gon to the intersections inside. Similarly for E, the n sides are counted as well as the segments inside the n- gon).

Hence the number of regions is $1 + \frac{1}{12}n(n-3)(n^2 - 3n + 8) - \frac{1}{24}n(n-1)(n-2)(n-3)$.

This simplifies to $\frac{1}{24}(n^4 - 6n^3 + 23n^2 - 42n + 24)$
$= \frac{1}{24}(n-1)(n-2)(n^2 - 3n + 12)$.

Q.856 The senior form has three classes, all with twenty students. Each student is acquainted with forty other seniors. Prove that there is at least one set of three mutual acquaintances, one of whom is in each class.

ANSWER. Since a student can have at most 19 acquaintances in his own class, he must have at least 21 in the other classes. Hence no student can have no acquaintances in one of the other classes. Let n be the smallest number of acquaintances of any student in a class other than his own. Let x, in class A, be a student with exactly n acquaintances in class B, one of whom is y. Now in class C, x has at least $(21 - n)$ acquaintances, and y has at least n acquaintances (because of the definition of n). As there are only 20 students in class C, there must be at least one student (z, say) in C who is acquainted with both x and y. Then x, y and z are a set of mutual acquaintances, one in each class.

Q.857 $a, b, c, \cdots, k$ is any set of n positive numbers.
Let $S = a + b + \cdots + k$ and $T = \frac{1}{a} + \frac{1}{b} + \cdots + \frac{1}{k}$.
Prove that $ST \geq n^2$.

ANSWER. It is convenient to change the notation, replacing $a, b, \cdots, k$ by a_i, $i = 1, 2, \cdots, n$. Then $S = \sum_{i=1}^{n} a_i$ and $T = \sum_{i=1}^{n} \frac{1}{a_n}$.

By coincidence this exercise appeared as part of Question 8 on the 4-unit H.S.C. mathematics paper in 1991. We give two workings, neither of which was supplied by any candidate in that examination, where the wording encouraged a proof by mathematical induction.

(1) If a and b are positive, and $s = a + b$, then $\frac{1}{a} + \frac{1}{b} \geq \frac{4}{s}$ since on multiplying through by sab the inequality is seen to be equivalent to $(a+b)^2 \geq 4ab$, which is immediately deducible from $(a-b)^2 \geq 0$. Thus the smallest value of $\frac{1}{a} + \frac{1}{b}$, (given s), is obtained when $a = b = \frac{s}{2}$.

Using this, if the positive numbers a_i are varied keeping S fixed, the smallest possible value of T must have all of the a_i equal; since if $a_k \neq a_j$ a smaller value of T would be obtained by replacing both of them with $\frac{a_k + a_j}{2}$.

Therefore $T_{\min}$ is obtained when $a_1 = a_2 = \cdots = a_n = \frac{S}{n}$

and $T_{\min} = \sum_{i=1}^{n} \left(\frac{n}{S}\right) = \frac{n^2}{S}$.

Therefore $ST \geq ST_{\min} = n^2$.

(2) The inequality $\frac{1}{a} + \frac{1}{b} \geq \frac{4}{a+b}$ proved above implies $(a+b)(\frac{1}{a} + \frac{1}{b}) \geq 4$. Hence $\frac{a}{b} + \frac{b}{a} \geq 2$ for positive a, b.

$$
\begin{aligned}
S.T &= \left(\sum_{i=1}^{n} a_i\right)\left(\sum_{j=1}^{n} \frac{1}{a_j}\right) \\
&= \sum_{k=1}^{n} \frac{a_k}{a_k} + \sum_{1 \leq i < j \leq n} \left(\frac{a_i}{a_j} + \frac{a_j}{a_i}\right).
\end{aligned}
$$

In the second sum there are clearly ${}^nC_2$ terms, and by the above calculation, none of them is less than 2.

$$\therefore \text{ S.T } \geq n + {}^nC_2 \times 2 = n + \frac{n(n-1)}{2} 2 = n^2.$$

Comment: $S/n = A$, the arithmetic mean of the numbers, and $n/T = H$, the "harmonic mean" of the numbers. Our working has proved the theorem that $A \geq H$ for any set of positive numbers. If instead one starts by quoting this theorem, the result follows immediately; a course of action taken by Jonathon Kong (Sydney Grammar School). The better known theorem that $A \geq G$ (the geometric mean) can be used to give another quick solution, since if $P =$ the product $\prod_{i=1}^{n} a_i$, the result follows from $S/n \geq \sqrt[n]{P}$ and $T/n \geq 1/\sqrt[n]{P}$.

Q.858 Let $a_1 = \sqrt{2}$; $a_2 = \sqrt{2}^{\sqrt{2}}$; $a_3 = \sqrt{2}^{\sqrt{2}^{\sqrt{2}}}$; $\cdots$; $a_n = \sqrt{2}^{\sqrt{2}^{\cdot^{\cdot^{\cdot^{\sqrt{2}}}}}}$ where there are n $\sqrt{2}$'s in the tower. Show that the list of numbers $a_1, a_2, \cdots, a_n, \cdots$ increases steadily, but that no matter how large n is, $a_n < 3$. Can you determine approximately how large a_n is when n is very big?

ANSWER. If $x > y$ and $a > 1$, $a^x = a^{x-y}a^y > 1.a^y$

Hence if $a_n > a_{n-1}$, $a_{n+1} = \sqrt{2}^{a_n} > \sqrt{2}^{a_{n-1}} = a_n$. Since $a_2 = \sqrt{2}^{\sqrt{2}} > \sqrt{2}^1 = a_1$, it follows that $a_3 > a_2$, then $a_4 > a_3$ and so on indefinitely.

Again if $a_n < 3$ then $a_{n+1} = \sqrt{2}^{a_n} < \sqrt{2}^3 = 2\sqrt{2} < 3$. Since $a_1 < 3$ it follows that $a_n < 3$ for all n.

We have shown that the list of numbers a_n is steadily increasing, but is "bounded above" by 3. Obviously the numbers must eventually get very close to some "limiting" value, ℓ. Taking n very large in $\sqrt{2}^{a_n} = a_{n+1}$, both a_n and a_{n+1} are very nearly equal to ℓ. In fact ℓ must satisfy $\sqrt{2}^{\ell} = \ell$. There are two obvious values of ℓ which satisfy this equation, viz. $\ell = 2$ and $\ell = 4$. Since all values of a_n are less than 3 they cannot get very close to 4. In fact they do get very close to 2. (If you know enough calculus, you can show that the function $f(x) = \sqrt{2}^x - x$ has just one stationary point; $f'(x) = 0$ only when $x = \frac{2}{\ell n\, 2} \ell n \left(\frac{2}{\ell n\, 2}\right)$. Hence there cannot be more than the two obvious solutions $x = 2$, or 4 of $f(x) = 0$.)

Q.859 In $\triangle ABC$, $AB = AC$. The bisector of $A\hat{B}C$ meets AC at D. If $BD + AD = BC$ find the angles of the triangle.

ANSWER. Rayman Yan (Randwick Boy's Technology High School), having first proved that $AB = \dfrac{BC}{2\cos\angle ABC}$ writes:-

$$\text{Let } \angle ABC = \angle ACB = 2\theta,\ AB = \frac{BC}{2\cos 2\theta}$$

$$\therefore \angle BAC = 180° - 4\theta,\ \therefore \angle ADB = 3\theta$$

In $\triangle ABD$, $\dfrac{\sin\theta}{AD} = \dfrac{\sin(180° - 4\theta)}{BD} = \dfrac{\sin 3\theta}{\frac{BC}{2\cos 2\theta}}$

$$\frac{\sin\theta}{AD} = \frac{\sin(180° - 4\theta)}{BD} = \frac{\sin\theta + \sin(180° - 4\theta)}{AD + BD} \quad \left(\text{since } \frac{a}{b} = \frac{c}{d} = \frac{a+c}{b+d}\right)$$

$$\therefore \frac{\sin\theta + \sin(180° - 4\theta)}{AD + BD} = \frac{2\sin 3\theta\cos 2\theta}{BC}$$

but $BC = BD + AD$ (as given). Therefore $\sin\theta + \sin(180° - 4\theta) = 2\sin 3\theta\cos 2\theta$.

$$\begin{aligned} \text{R.H.S.} &= 2\sin 3\theta\cos 2\theta \\ &= 2 \times \left(\frac{1}{2}[\sin 5\theta + \sin\theta]\right) \quad \left(\text{since } \sin A\cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]\right) \\ &= \sin 5\theta + \sin\theta \end{aligned}$$

$$\therefore \sin(180° - 4\theta) + \sin\theta = \sin 5\theta + \sin\theta$$

$$\sin(180° - 4\theta) = \sin 5\theta$$

$$180° - 4\theta = 5\theta$$

$$9\theta = 180°$$

$$\theta = 20°$$

Therefore the angles of the triangle are $40°, 40°, 100°$.

Q.860 Two congruent circles intersect at AB. Point P lies on one circle, and Q on the other. M, N are the feet of the perpendiculars from A to the lines BP and BQ respectively. Prove that the mid points of AB, PQ and MN are collinear.

ANSWER. The three figures cover the possibilities that P and Q both lie on the major arcs on AB (Fig. 1) or both on the minor arcs (Fig. 3), or one on a major arc, one on a minor arc (Fig. 2).

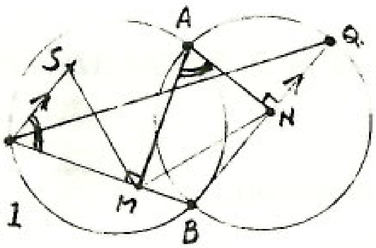


Fig 1

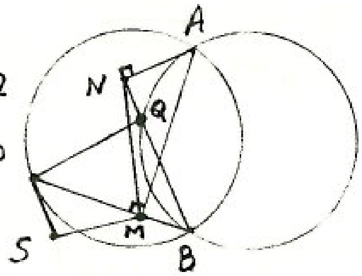


Fig 2

Since $\angle ANB$ and $\angle AMB$ are right angles, the circle on diameter AB passes through both M and N. Thus the perpendicular bisector of the chord MN passes through the centre of that circle, the mid point of AB.

Construct PS equal and parallel to NQ, so $PNQS$ is a parallelogram and the mid point X of PQ is also the mid point of the other diagonal NS. We will show that $\angle SMN$ is a right angle, so that the circle on diameter SN passes through M. Then it will follow that X too lies on the perpendicular bisector of MN, and the desired result is established.

Since AB is a common chord of congruent circles, the angles $\angle APB$ and $\angle AQB$ subtended at the circumferences are either equal, or supplementary, and in every case $\angle APM = \angle AQN$

$$\therefore \frac{MP}{PS} = \frac{MP}{NQ} = \frac{AM \cot \angle APM}{AN \cot \angle AQN} = \frac{MA}{AN}.$$

Using this we can now deduce that $\Delta MPS \sim \Delta MAN$ since the included angles $\angle MPS$ and $\angle MAN$ are equal in every figure. (In figs. 1 and 3, both angles are supplementary to $\angle MBN$, the first because $PS \| BQ$, the second because $MBNA$ is a cyclic quadrilateral. In fig. 2

$$\angle MPS = \angle MBN \text{ (alternate angles)}$$
$$= \angle MAN \text{ (since } MBNA \text{ is cyclic).).}$$

From the similarity of these triangles, $\angle PMS = \angle AMN$. It follows that $\angle SMN = \angle AMP = 1$ right angle, and the proof is complete.

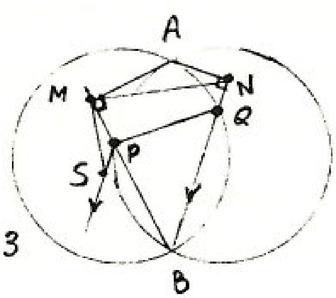


Fig 3