# HONEYBEES AND ISOPERIMETRICS

**Ray Popple***

When a honeybee constructs its combs to store its honey, it does so in such a way as to store the maximum amount of honey using the minimum amount of wax.

The honeybee has solved this problem by constructing its combs in the shape of regular hexagons. Why regular hexagons?

The bees require the shape of the combs to tessellate the plane. That is they require there be no crevices into which honey can disappear. An arrangement of circular combs, while encasing the largest area would be wasteful.

In fact there are only three regular polygons that tessellate the plane: equilateral triangles, squares and regular hexagons. (See diagram 1).

The reason for this is almost obvious. The interior angles of regular triangles, quadrilaterals and hexagons are 60°, 90° and 120° respectively, and 360° is an integral multiple of each of these. Hence the number of cells about a point will be that integer. On the other hand regular pentagons do not tessellate and nor do $n$-gons where $n > 6$. This is because regular polygons of more than 6 sides have interior angles between 120° and 180° and there is no number between 120° and 180° which when multiplied by an integer is 360°. Therefore the only tessellations available to the honeybee are squares, equilateral triangles and regular hexagons.

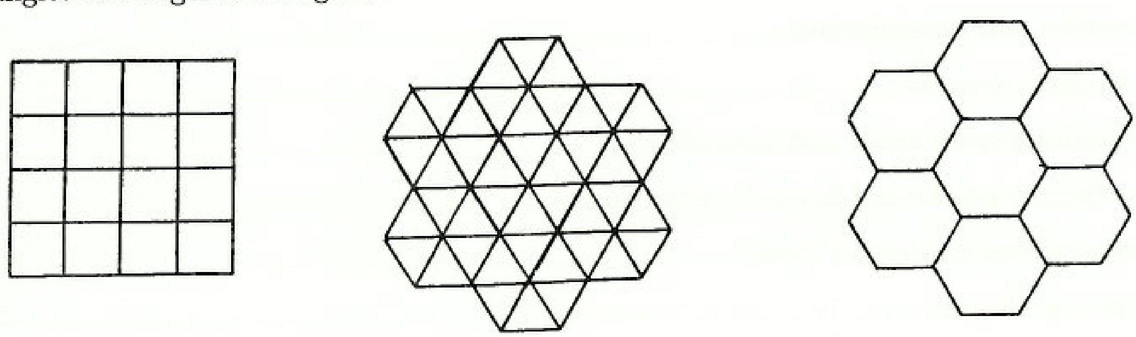

DIAGRAM 1

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Now consider an equilateral triangle, a square and a regular hexagon that have the same perimeter (and hence use the same volume of wax). Such figures are said to be isoperimetric.

Let the perimeter of each be $S$. Then:

Area of equilateral triangle is $A_t = \frac{1}{2}\left(\frac{S}{3}\right)^2 \sin 60° = \frac{\sqrt{3}}{36}S^2$

Area of square is $A_s = \frac{S^2}{16}$

Area of hexagon is $A_h = 6.\frac{1}{2}.\frac{S}{6}\frac{S}{12}\tan 60° = \frac{\sqrt{3}}{24}S^2$.

A simple comparison shows that

$$A_t < A_s < A_h.$$

Thus the regular hexagon has the largest area of the three isoperimetric shapes that tessellate. In fact, the regular hexagon is approximately 15.5% better than the square and 50% better than the triangle, when it comes to storing honey.

These facts about honeycombs were first noted by Pappus of Alexandria in about 320 in his famous work with the title **Collection**. The mathematical activity of bees has also been investigated by the great astronomer Johannes Kepler and the 18th Century French mathematician Jacques-Philippe Miraldi.

These results are closely related to the solution of the isoperimetric problem as stated by Pappus. **"Of all regular plane figures having equal perimeters, that which has the greater number of sides is always the greater (in area), and the greatest of them all is the circle having its perimeter (circumference) equal to them."**

While this may seem obvious its verification produces some interesting mathematics. We break our argument up into six points.

**1.** "Triangles having the same base with vertices that lie on a line parallel to that base have equal areas."

This is Proposition 37 in Euclid's **Elements** and we leave its proof as an exercise.

**2.** "In the set of all triangles having the same base and equal areas, the isosceles triangle has the least perimeter."

**Given:** $\triangle ABC$ is isosceles with $AB = AC$, line $k \parallel$ line $\ell$. $Y$ is some point on $k$ distinct from $A$ such that area $\triangle ABC \equiv$ area $\triangle YBC$. (See diagram 2.)

**Prove:** The perimeter of isosceles $\triangle ABC <$ the perimeter of $\triangle YBC$.

**Construction:** Extend $BA$ to $X$ such that $BA = AX$. Join $XY$.

**Proof:**

$$X\widehat{A}Y = A\widehat{B}C \text{ (corresponding)}$$

$$\therefore X\widehat{A}Y = B\widehat{C}A \text{ } (\triangle ABC \text{ is isosceles})$$

$$\therefore X\widehat{A}Y = C\widehat{A}Y \text{ (alternate)}$$

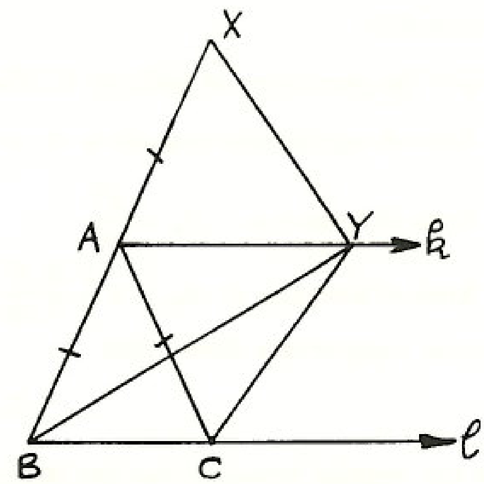



DIAGRAM 2

Hence $\triangle AXY \equiv \triangle ACY$ $(SAS)$.

$\therefore YX = YC$ (corresponding sides). But in any triangle the sum of two sides is greater than the third side.

$$\therefore \; BY + YX > BX$$

$$\text{i.e., } BY + YC > BA + AX$$

$$\text{i.e., } BY + YC > BA + AC$$

$$\therefore \; BA + AC + CB < BY + YC + CB$$

Thus the perimeter of the isosceles triangles is less than the perimeter of any other triangle with the same base and area.

**3.** Now we consider the case of all triangles of equal perimeter (isoperimetric) having the same base. We show that the one with the greatest area is isosceles.

**Given:** Isoperimetric triangles $ABC$ and $XBC$, and $AB = AC$.

**Prove:** Area $\triangle ABC >$ Area $\triangle XBC$.

**Construction:** Draw the altitude $AY$ of $\triangle ABC$. Let $Z$ be the point on $AY$ such that $ZX \| BC$. Join $ZB$ and $ZC$. (See diagram 3.)

**Proof:** $\triangle ABC$ is isosceles and hence $AY$ is the perpendicular bisector of $BC$.

Therefore $ZB = ZC$, and hence $\triangle ZBC$ is isosceles. Now triangle's $ZBC$ and $XBC$ lie on the same base with the other vertices on a line parallel to the base, hence by result 1, above

$$\text{Area } \triangle ZBC = \text{ Area } \triangle XBC.$$

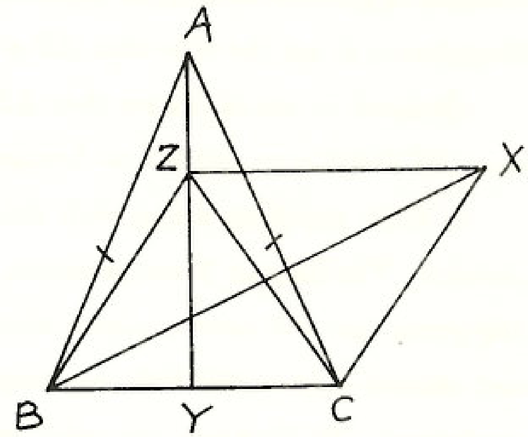



DIAGRAM 3

Therefore by result 2, above

$$BX + XC > BZ + ZC$$

But triangles $ABC$ and $XBC$ are isoperimetric hence

$$AB + BC + CA = XB + BC + CX$$

i.e., $AB + CA = XB + CX$

Therefore $AB + CA > BZ + ZC$. But $AB = CA$ and $BZ = ZC$. Hence $AB > ZB$ and $AY > ZY$

$\therefore$ Area $\triangle ABC >$ Area $\triangle ZBC =$ Area $\triangle XBC$.

We shall now extend this investigation to polygons and in particular consider the case of the hexagon.

**4.** In the set of isoperimetric polygons of the same number of sides, the one with the greatest area must be equilateral.

We'll only consider the case for six sides.

**Given:** $ABCDEF$, a hexagon of greatest area amongst all isoperimetric hexagons.

**Prove:** $ABCDEF$ is equilateral, i.e., $AB = BC = CD = DE = EF = FA$.

**Construction:** Join $BF$ and construct $\triangle BA'F$ so that $BA' = FA'$ and $\triangle BA'F$ is isoperimetric with $\triangle BAF$. (See diagram 4.)

**Proof:** Assume $AB \neq AF$. Now as $\triangle BA'F$ is isosceles, Area $\triangle BA'F >$ Area $\triangle BAF$ (result 3)

$\therefore$ The area of hexagon $A'BCDEF >$ area hexagon $ABCDEF$.

But triangles $BA'F$ and $BAF$ are isoperimetric.

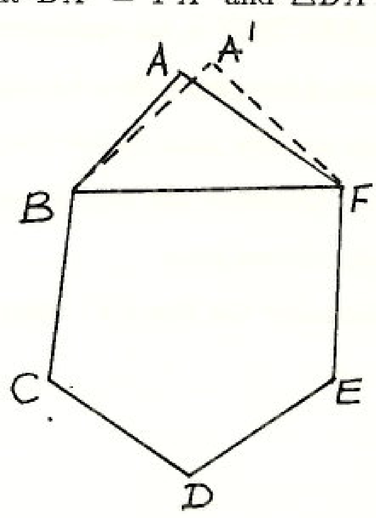



DIAGRAM 4

i.e. $A'BCDEDF$ and $ABCDEF$ are isoperimetric. Therefore $ABCDEF$ is NOT the hexagon of greatest area of all isoperimetric hexagons. This contradicts our original data. Therefore it is not the case that $AB \neq AF$. Therefore $AB = AF$.

Similarly it can be shown that $AB = BC = CD$ etc.

i.e. $ABCDEF$ is an equilateral hexagon.

Is there anything wrong with this argument? Yes, there is but the fallacy is not so obvious. The famous Swiss geometer Jacob Steiner fell into the same trap more than 100 years ago. We have assumed that **there always is a polygon of maximal area** and showed that **under this assumption** (namely if such a polygon exists) it must be equilateral. To illustrate the point we show by a similar fallacious argument that 1 is the largest integer. Why? If this is not so, assume $n > 1$ is the largest integer. Then $n^2 > n$ gives a contradiction. There is no contradiction if $n = 1$ which is therefore the largest integer – a plain absurdity. Of course what we have shown is that **if there exists a largest integer** then it must be 1. In the case of polygons the assumption that there exists one with largest area is correct but we must omit the proof which is quite subtle.

**5.** We now extend this to show that the isoperimetric polygon of greatest area is not only equilateral, but regular. Once again we shall prove it for a hexagon, since a truly rigourous proof is beyond the level of this article.

We need two preliminary results.

(a) If two sides of a triangle are given, the triangle of greatest area is formed when the two sides are perpendicular. We leave it to the reader to find a proof which doesn't use trigonometry.

(b) If $ABCDEF$ is an equilateral hexagon that has the greatest area of all isoperimetric equilateral hexagons, then for any vertex, there is another vertex such that the line joining the two vertices must divide the area of $ABCDEF$ into two quadrilaterals of equal area.

i.e. Consider the equilateral hexagon that has the greatest area of all isopermetric equilateral hexagons.

Consider the line $CF$. Assume area $ABCF >$ area $EDCF$.

Now consider what happens when $ABCF$ is reflected in the line $FC$. Let $A$ and $B$ have images $A'$ and $B'$ in the reflection. (See diagram 5.) But reflection does not change the area of a figure. (We say area is invariant under reflection). Similarly length is invariant.

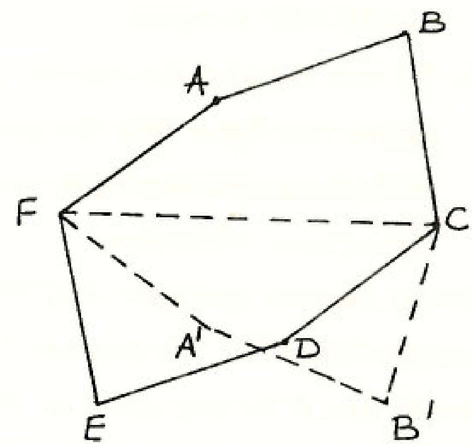



DIAGRAM 5

Therefore the area $ABCB'A'F >$ area $ABCDEF$ and $ABCB'A'F$ and $ABCDEF$ are isoperimetric. This is therefore a contradiction. Similarly the assumption that area $ABCF <$ area $EDCF$ leads to a contradiction.

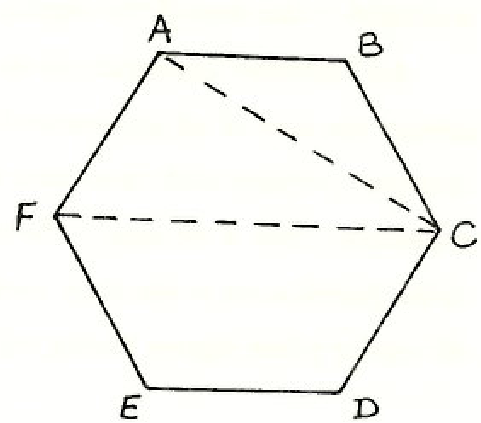
$$\therefore \text{ area } ABCF = \text{ area } EDCF.$$

i.e., $FC$ divides the hexagon into two quadrilaterals of equal area.

**Given:** $ABCDEF$ has the greatest area of all isoperimetric equilateral hexagons.

**Prove:** $ABCDEF$ is regular, i.e. $ABCDEF$ can be inscribed in a circle. See diagram 6.

**Construction:** Join $AC$ and $CF$.




**Proof:** Triangle $ACF$ must have the maximum area of all triangles that can be formed with $AC$ and $AF$.

Therefore by (a) above $C\widehat{A}F = 90°$. Why? If $C\widehat{A}F$ is to be 90°, then by making $C\widehat{A}F = 90°$ without altering $AC$ and $AF$, we would increase the area of $\triangle ACF$ without altering the size of $\triangle ABC$.

The area of $\triangle ABC$ cannot be altered as its three sides are fixed, but since the area of $\triangle ACF$ would be increased, the area of quadrilateral $ABCF$ would be increased. But this is impossible by (b) above since $ABCDEF$ would not have the greatest area.

$\therefore C\widehat{A}F = 90°$.

Hence $A$ lies on the semicircle, with diameter $CF$.

Similarly $B, E$ and $D$ all lie on the circle with diameter $\dot{C}F$.

$\therefore ABCDEF$ can be inscribed in a circle with diameter $CF$.

$\therefore ABCDEF$ is regular.

**6.** We now consider the original proposition of Pappus that for regular isoperimetric figures, the figure with the greater number of sides has the greater area. We will verify this proposition for the specific case of a regular pentagon and a regular hexagon having equal perimeters.

**Given:** Regular pentagon $ABCDE$ and regular hexagon $PQRSTU$. (See diagram 7.)

**Proof:** Let $Z$ be any point on $DC$. Therefore $ABCZDE$ can be considered as a hexagon. Therefore $ABCZDE$ is a non-regular hexagon. Therefore by results it is a hexagon of area less than the area of a regular hexagon of equal perimeter.

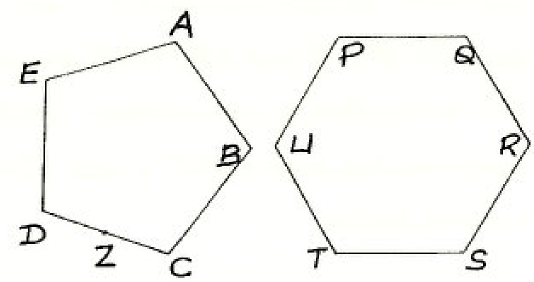



DIAGRAM 7

i.e. Area $ABCZDE <$ Area $PQRSTU$.

i.e. Area $ABCDE <$ Area $PQRSTU$.

i.e. If a regular pentagon and a regular hexagon are isoperimetric, the area of the regular pentagon $<$ the area of the regular hexagon.

An inductive argument can be used to show that the greater the number of sides the greater the area of all isoperimetric regular polygons. It follows therefore that there is no isometric polygon with maximum area (just as there was no largest integer in our earlier argument). But if we regard the circle as a regular "polygon" with infinitely many sides (some justification is certainly needed here) we arrive to the conclusion of Pappus that of all regular plane figures having equal perimeters the circle has the greatest area.