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THE GOLDEN NUMBER

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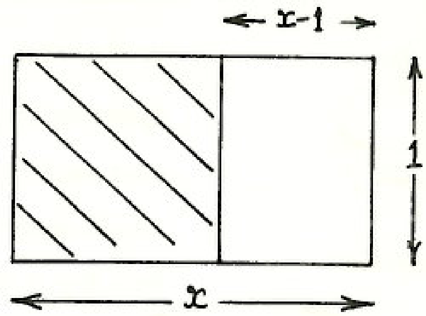
Without any doubt π is the most famous of all real numbers, it appears throughout all branches of mathematics. Pi is of course irrational, that is it cannot be expressed in the form $p/q, p, q \in \mathbb{Z}, q \neq 0$. Another irrational number which receives less publicity but which has countless surprising and elegant properties, particularly in the field of natural growth is the golden number which we shall denote by ϕ (pronounced Phi).

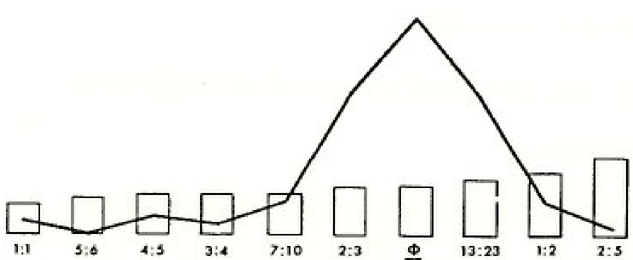
Definition of ϕ:

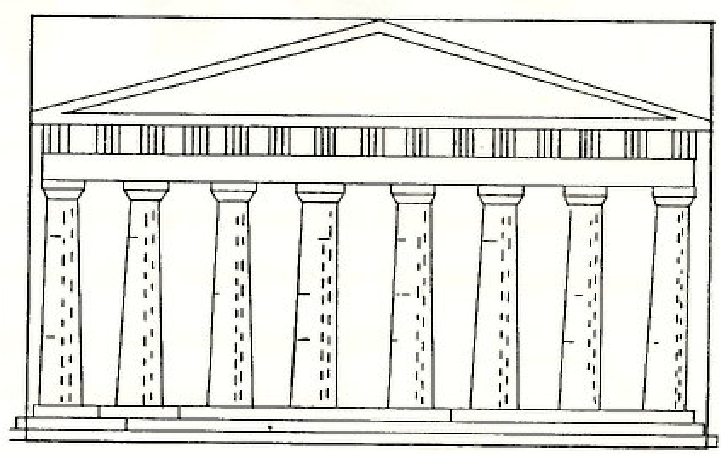
Suppose we wish to partition a rectangle into a square and a smaller rectangle in such a manner that the smaller rectangle has exactly the same proportions as the larger.

$$\frac{x}{1} = \frac{1}{x-1} \rightarrow x(x-1) = 1$$
$$\rightarrow x^2 - x - 1 = 0$$
$$\rightarrow x = \frac{1 \pm \sqrt{1+4}}{2}$$
$$= \frac{1 \pm \sqrt{5}}{2}$$



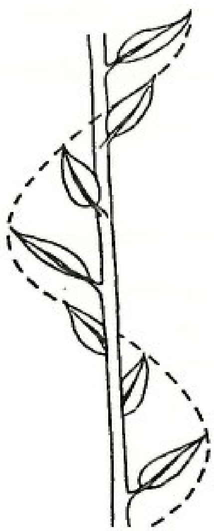
We discard $\frac{1-\sqrt{5}}{2}$ since it is negative, and define $\phi = \frac{1+\sqrt{5}}{2} \doteqdot 1.618$ (the Golden Number or Golden Ratio). We call any rectangle with the proportions $1 \times \phi$ a Golden Rectangle. Various psychological studies have revealed that of all rectangles the Golden Rectangle is most pleasing to the eye and aesthetically attractive. In the graph below a large sample was tested for personal preferences in rectangles, with the golden rectangle clearly dominating.





The Parthenon at Athens, built in the fifth century B.C., one of the world's most famous structures. While its triangular pediment was still intact, its dimensions could be fitted almost exactly into a Golden Rectangle, as shown above. It stands therefore as another example of the aesthetic value of this particular shape.

The ancient Greeks were not only aware of ϕ and its various properties, but also used the golden rectangle in their architecture. The golden ratio is mentioned in Euclid's **Elements** and has fascinated scholars for thousands of years. When encountering rectangles (sheets of paper, playing cards, books etc) it is interesting to check how close they are to the golden rectangle.



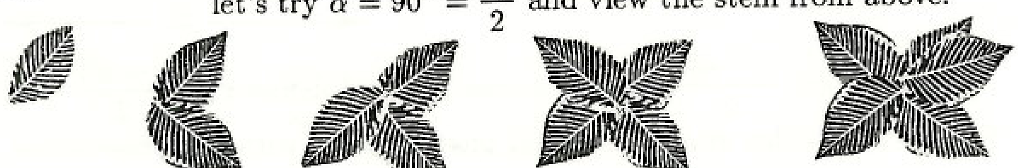
Phyllotaxis

Properties of ϕ

(1) **Leaf Growth** (Phyllotaxis)

Consider the stem of a plant which is growing vertically and sprouting the occasional leaf. We assume that each successive leaf is produced at a constant angle α from its predecessor.

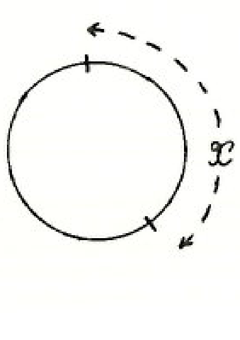
Question: Which angle α spreads the leaves around so that each leaf receives maximum exposure to the sun, i.e. which angle best utilises the space around the stem? Well let's try $\alpha = 90° = \frac{\pi^c}{2}$ and view the stem from above.



This is O.K. for the first few, but after a while the 5th leaf totally blocks out the first etc. How about $\alpha = 45°$?

Once again we are getting interference and empty space is being wasted. Furthermore after 9 leaves we will encounter the same problem as in $\alpha = 90°$.

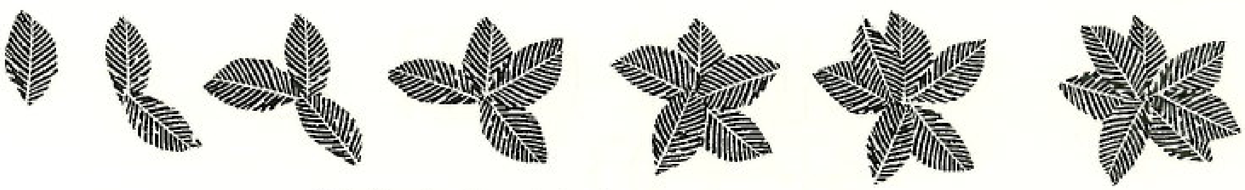
It can be shown (the proof is quite difficult) that the plant's optimal strategy is to constantly divide the circumference in the golden ratio.



What angle does this correspond to? Well let's assume a circle of radius 1. Then the circumference is 2π. We demand $\frac{2\pi - x}{x} = 1.618$

$$\to 2\pi - x = (1.618)x \to 2.618x = 2\pi \to x = \frac{2\pi}{2.618}$$

and using $\ell = r\theta \to \ell = \theta$ we have $\theta = \dfrac{2\pi}{2.618}$ radians $\doteqdot 137\frac{1}{2}^{\circ}$. This is called the golden angle and it can be seen that this value of α does a nice job in spreading the leaves around. In fact it spreads them around in the best possible way.



It is fascinating to look at real plants and see how the golden angle is used to distribute leaves and branches.

Fibonacci Numbers

The Fibonacci sequence is defined very differently from the golden ratio, however they are closely related and together are the principal tools in the analysis of natural growth.

Rabbits

Fibonacci (an Italian mathematician born about 1175) first considered this special sequence in describing the total population of a colony of rabbits. Suppose we begin with 1 pair of new-born rabbits, each rabbit takes 1 month to mature and thereafter each pair produces a pair of rabbits monthly. We denote by F_n the number of **pairs** of rabbits at the end of the nth month. The situation is best represented by a tree diagram, where B represents a new-born rabbit pair, 0 a mature (i.e. breeding) pair.

$F_6 = 8$

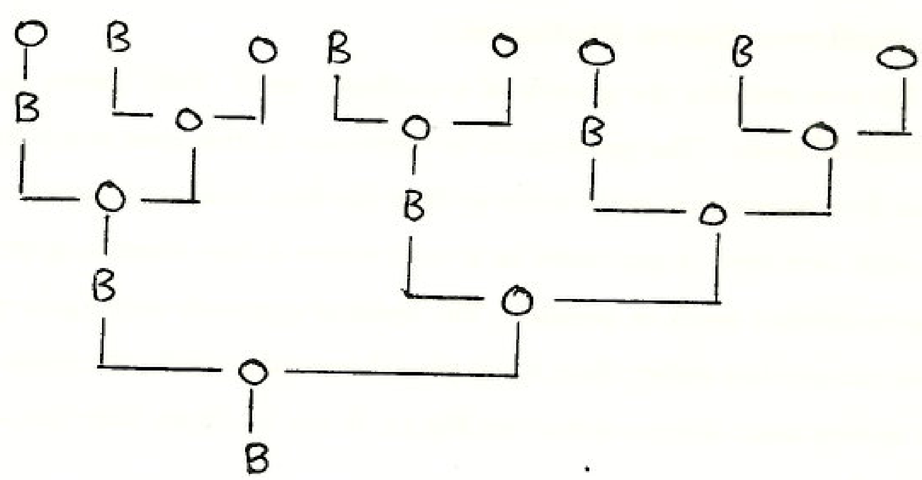
$F_5 = 5$

$F_4 = 3$

$F_3 = 2$

$F_2 = 1$

$F_1 = 1$



We obtain the Fibonacci sequence $1, 1, 2, 3, 5, 8, 13, 21, 34 \cdots$ which can be described as

$$F_1 = 1, \; F_2 = 1, \; F_n = F_{n-1} + F_{n-2} \quad n = 3, 4, 5, \cdots$$

Note that the Fibonacci sequence is neither an A.P. nor a G.P. so your H.S.C. formulae are no good here. The formulae for the nth term is given by

$$\begin{aligned} F_n &= \frac{1}{\sqrt{5}} \left\{ \left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right\} \\ &= \frac{1}{\sqrt{5}} \left\{ \phi^n - \left(-\frac{1}{\phi} \right)^n \right\} \end{aligned}$$

That is, the nth Fibonacci number basically depends on the golden number.

Question: How can the Golden Number be extracted from the Fibonacci sequence?

Well $\frac{3}{2} = 1.5, \frac{5}{3} = 1.66, \frac{8}{5} = 1.6, \frac{13}{8} = 1.625$ and it looks like $\lim_{n \to \infty} \frac{F_{n+1}}{F_n} = \phi$.

Proof: Let $x = \lim_{n \to \infty} \frac{F_{n+1}}{F_n} = \lim_{n \to \infty} \frac{F_n + F_{n-1}}{F_n} = \lim_{n \to \infty} 1 + \frac{F_{n-1}}{F_n} = \lim_{n \to \infty} 1 + \frac{1}{F_n / F_{n-1}} =$ $1 + \frac{1}{x} \to x = 1 + \frac{1}{x} \to x^2 = x + 1 \to x^2 - x - 1 = 0 \to x = \phi$.

Fibonacci numbers spring up throughout nature. Try counting the number of petals in a flower or the number of leaves in a cluster, how many legs does a starfish have; an octopus? We finish with another remarkable connection between Fibonacci numbers and ϕ to be found in the head of a sunflower.

The Sunflower (Spiral Phyllotaxis)

We now consider the growth of a sunflower head. Fully grown the head consists of hundreds of seeds. The problem to be overcome in this case is not access to light, but rather how to generate extra seeds so that the final head is as tightly packed as possible and each new seed is generated in a spot where it has room to grow, i.e. as far away from established seeds as possible. The optimal approach once again involves the golden angle, except that rather than being placed around a circle, the seeds are positioned via that golden angle along a spiral (see Fig.1). It can be shown that this is the most efficient approach.

What is remarkable is that two sets of secondary spirals appear, one running clockwise, the other anti-clockwise (see Fig.2). Counting up the number of secondary spirals we find there are 8 and 13, consecutive Fibonacci numbers!! This is always the case. Next time you see a sunflower, daisy or even the base of a pineapple, count up the Fibonacci spirals. (See Fig.3).

Fibonacci numbers and the golden number run as a unifying thread through the theory of growth. It is important to realize however that plants are not mathematicians. They do not work out where to place the next seed or leaf, it is simply put where there is most room. The amazing structure and elegant mathematics springs from this fundamental law of growth.

UNEXPECTED EXAMINATIONS – A POSTSCRIPT

David Angell

It is well known that life imitates art. The very day that I handed in my article for the previous edition of **Parabola**, a lecturer here at UNSW came to talk to me about planning the assignment schedule for the course he was to give in second session. His intention was to set six short assignments to be written and brought to class; but only three of these were to be collected for marking, and the students were not to be told in advance which three these were to be. If this sounds familiar, try to think through the consequences before reading further.

It would seem that the sixth assignment must be collected. For otherwise, three of the first five will be collected, and then lazy students (yes, there are some) won't bother to do the sixth. Likewise, the first must be collected; otherwise, after the fourth, students will know that there is only one left, which must be the sixth, and therefore it is not necessary to do the first. In the same way, the fourth assignment must be collected. The student who has thought all this through, therefore, knows that it is not essential to do any of the first three assignments as they definitely won't be collected. On the other hand, the student who has read the last edition of **Parabola** will realise that ...

What would you do?