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SOME FAMOUS POINTS AND LINES OF A TRIANGLE

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A triangle is one of the simplest configurations we can draw in the plane. Nevertheless, mathematicians have been fascinated by its properties for centuries. New and unexpected discoveries about the triangle have been made right into our times. Studying some of these facts we not only find a richness of material, but also absorb a variety of logical reasoning which contribute to the beautiful structure of geometry.

It is well-known that the perpendicular bisector of the line-segment AB contains all the points in a plane which are equidistant from A and B, i.e. it is the locus of all such points. Considering the $\triangle ABC$, let the perpendicular bisectors of the sides BC and of AB meet at a point O. Then O is equidistant from B and C and also from A and B, i.e. it is equidistant from all 3 vertices. So O is the centre of the circle that we can draw to pass through A, B and C. O is the circumcentre of the triangle. Also, O is equidistant from A and C, therefore the third perpendicular bisector has to pass through the same point O.

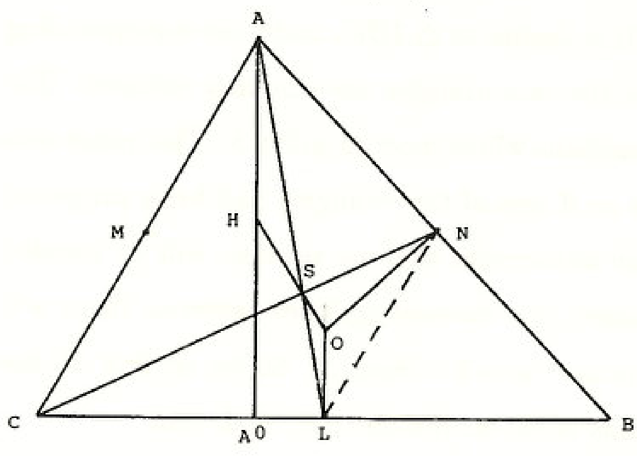

Figure 1

In figure 1) A^0 represents the foot of the perpendicular from A to BC, L, M, N represent the midpoints of the sides they are on. AL, CN are two medians, intersecting at S. It is well-known that NL is parallel to AC and equal in length to $\frac{1}{2}AC$. Therefore

$$\triangle SNL \;|||\; \triangle SCA \text{ (equal angles)} \qquad (1)$$

and $AS \div SL = AC \div NL = 2:1$.

If we had drawn BM instead of CN, the same argument would hold, BM would intersect AL at the same point S, which is 2/3 of AL from the vertex. This proves that the 3 medians are concurrent at S, called the centroid of the triangle. S has the interesting property that if we place equal masses in the vertices of the triangle, their centre of mass

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will be at S. Now joins O to S and extend it to meet the altitude AA^0 at a point H. Then

$$\triangle AHS \parallel\!| \triangle LOS \text{ (equal angles)} \tag{2}$$

and

$$HS \div SO = AS \div SL = 2:1.$$

This means that the point H on the extension of OS, is the same for all 3 altitudes. This proves that H is the point where the 3 altitudes must be concurrent, called the orthocentre of the triangle. The line OS, that contains the circumcentre, the centroid and the orthocentre of the triangle, is called the Euler line of the triangle. It is important also to note that

$$AH \div OL = 2 \div 1.$$

We should remark that Euler was one of the greatest mathematicians in the history of mathematics. Born in Switzerland, in 1707, he lived a great part of his life in St. Petersburg, at the time of Catherine the Great and died there in 1783.

Let us consider now the triangle LMN. It is similar to $\triangle ABC$, and their corresponding sides are parallel to each other. We say that the two triangles are similarly situated. The lines joining corresponding vertices are the medians which meet at point S. This point acts as the centre of similarity. The situation is as if one of the triangles had been projected from S to the other triangle. Each point or distance in the one triangle will be parallel to its corresponding pair in the other triangle, the connecting lines between them will pass through S and they will have the constant ratio 2:1 from S. So let us find F, the circumcentre of $\triangle LMN$ (see Figure 2). It has to be on the line OS, such that

$$OS = SF = 2:1.$$

But OS is the Euler line and we know that $OS = \frac{1}{3}OH$, so

$$\begin{aligned} OF = OS + SF &= \frac{1}{3}OH + \frac{1}{2}OS \\ &= \frac{1}{3}OH + \frac{1}{6}OH = \frac{1}{2}OH, \end{aligned}$$

i.e. F is the midpoint of OH, and is the centre of the circle that passes through the three midpoints of the sides of the triangle. We are going to prove that it passes through 6 other remarkable points, and for that reason it is called the nine-point circle. Join L to F and extend it to meet AA^0 in X. Then

$$\triangle FLO \equiv \triangle FXH$$

as $HF = FO$ and all the angles are equal. In particular

$$FL = FX,$$

but FL is the radius of the circle, therefore X is a point on the circle. But we also know that

$$HX = OL = \frac{1}{2}AH,$$

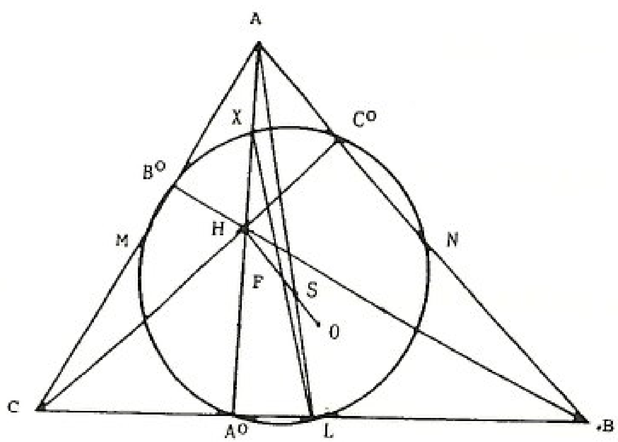

Figure 2

so the circle cuts the altitude from A at a point halfway between A and H. Similarly it has to cut the other two altitudes in their corresponding points, this gives another 3 points to the 3 mid-points. We realize also that LX is a diameter of the circle, and

$$\angle XA^0L = 90^0,$$

so A^0 is also a point on the circle.

Therefore we see that the 3 footpoints of the altitudes are also points on this remarkable circle, this makes 9 points.

We denoted the centre of the circle F in honour of Feuerbach, a German mathematician who described it in 1822.

The accompanying Figure 1 and Figure 2 show the triangles as acute-angled. It is interesting to follow up the case of right-angled and obtuse-angled triangles, the figures will look different, but these theorems remain valid.

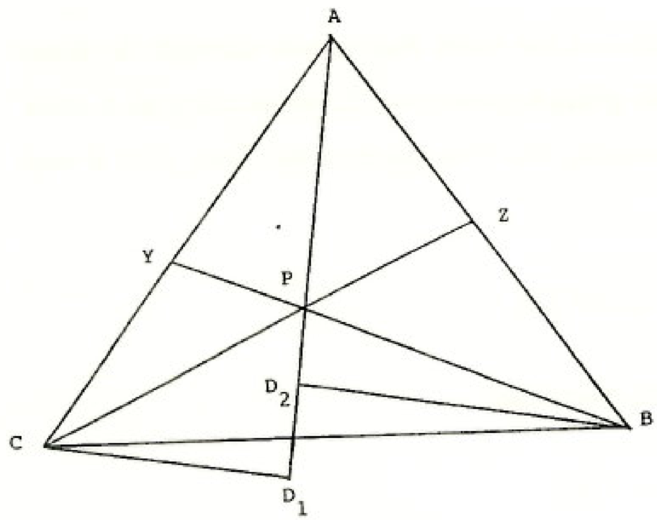

Figure 3

Consider now 3 lines, AX, BY, CZ, each drawn in $\triangle ABC$, from a vertex to a point on the opposite side of the triangle. What is the condition these lines have to satisfy if they are to be concurrent at a point P? Assume that they are in fact concurrent, then they divide $\triangle ABC$ into 3 smaller triangles: CPB, BPA and APC. Look at triangles APC and APB (Figure 3). They have a common side, AP, therefore their areas will be in the same ratio as $CD_1 \div BD_2$, i.e. as the ratio of the altitudes drawn to AP in the two triangles. But, by similar triangles, we can easily see, that

$$CD_1 \div BD_2 = CX : XB$$

so

$$\text{area } \triangle APC \div \text{ area } \triangle APB = CX \div XB \qquad 1)$$

Similarly

$$\text{area } \triangle CPB \div \text{ area } \triangle CPA = BZ \div ZA \qquad 2)$$

and

$$\text{area } \triangle APB \div \text{ area } \triangle BPC = AY \div YC \qquad 3)$$

Multiplying these 3 equations, we get

$$\frac{\text{area } \triangle APC}{\text{area } \triangle APB} \times \frac{\text{area } \triangle CPB}{\text{area } APC} \times \frac{\text{area } \triangle APB}{\text{area } \triangle CPB} = \frac{CX}{XB} . \frac{BZ}{ZA} . \frac{AY}{YC} = 1 \qquad 4)$$

so this condition is necessary if the 3 lines are concurrent. We can prove that it is also sufficient. Assume that we have 3 lines which satisfy condition 4) but which are not concurrent. Let P be the intersection of AX and BY, and assume that CP meets AB in a point Z_1. Then, the 3 lines AX, BY and CZ_1 will be concurrent and therefore satisfy the condition

$$\frac{CX}{XB} \cdot \frac{BZ_1}{Z_1A} . \frac{AY}{YC} = 1 \qquad 5)$$

But we have assumed that

$$\frac{CX}{XB}\cdot\frac{BZ}{ZA}\cdot\frac{AY}{YC} = 1 \qquad 4)$$

Clearly, 4) and 5) can only be consistent if $Z = Z_4$, i.e. the 3 lines are concurrent.

This theorem was discovered by the Italian mathematician Ceva, in 1678. A line joining a vertex of a triangle to a point on the opposite side is therefore often called a Cevian.

A similar theorem holds if 3 points on the lines that form a triangle are collinear. One of the points is then necessarily on the extension of one of the sides. (See Figure 4). Draw CP to be parallel to side AB.

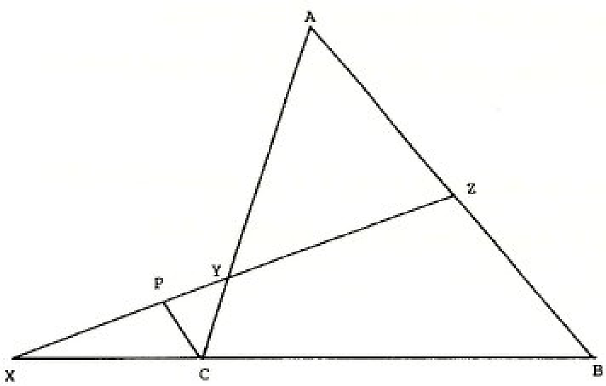

Figure 4

Then, by similar triangles

$$\frac{AY}{YC} = \frac{ZA}{PC} \qquad 6)$$

$$\frac{CX}{XB} = -\frac{PC}{BZ} \qquad 7)$$

(we put here a negative sign to indicate that CX and XB are directed oppositely) and

$$\frac{BZ}{ZA} = \frac{BZ}{ZA} \qquad 8)$$

Multiplying equations 6), 7) and 8) we get:

$$\frac{AY}{YC}\cdot\frac{CX}{XB}\cdot\frac{BZ}{ZA} = \frac{ZA}{PC}\cdot\left(-\frac{PC}{BZ}\right)\cdot\frac{BZ}{ZA} = -1 \qquad 9)$$

This is a necessary condition that the 3 points X, Y, Z should be collinear. We can prove that it is also sufficient condition in a way similar to the proof in Ceva's theorem. Try it!

This theorem was first announced by Menelaus, in A.D. 100.

It is a surprising fact that two theorems as similar in content and form as Ceva's and Menelaus' theorem have been discovered more than a thousand years apart. To understand this we have to look at historical facts. Menelaus lived and worked in Alexandria. This famous city was founded in 332 B.C. by Alexander the Great. After his death the city became part of Egypt and soon developed into a great centre of knowledge. Euclid's "Elements of Geometry" was written in Alexandria in or around 300 B.C. This book is still today the foundation of our geometrical thinking. Alexandria had a library, which contained all the knowledge of the ancient world. This was invaluable at a time when printing was not yet invented. Sadly, this most famous library was destroyed by 641, at the time of the arab expansion into Northern Africa. During the next thousand years the history of Europe is mainly the history of various wars. Like Menelaus' theorem much ancient knowledge was forgotten and had to be reinvented during the time of the Renaissance or after.

Problems

1) What can we say about the triangle if the ninepoint circle is tangent to one of the sides?
2) Describe the Euler line of the triangle formed by the midpoints of the sides.
3) Given the midpoint of one side, the circumcentre and the centroid describe how to construct the triangle.
4) In $\triangle ABC, Y$ is a point on AB, Z is a point on AC such that YZ is parallel to BC. Show that BY and CZ intersect at a point P which lies on the median AM.

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