Parabola Volume 28, Issue 2 (1992)

WE MET BY CHANCE

A Probability Problem with Infinitely Many Outcomes

by Peter Brown

The last question on the Westpac Mathematics Competition, Senior Division, 1989 reads:

> "Two forgetful friends agree to meet in a coffee shop one afternoon but each has forgotten the agreed time. Each remembers that the time was somewhere between 2pm and 5pm. Each decides to go to the coffee shop at a random time between 2pm and 5pm, wait half an hour and leave if the other doesn't arrive. What is the probability that they meet?"

A slightly simplified version is the problem:

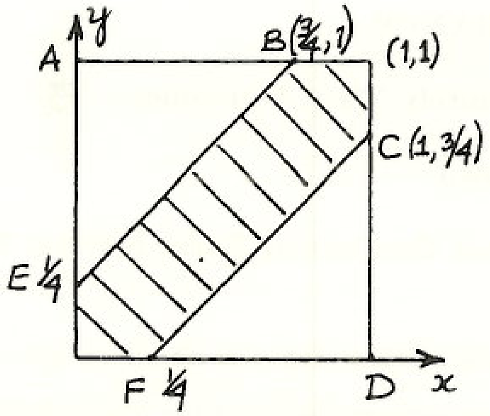
> "Two people agree to meet at a given place between noon and 1pm. By agreement, the first to arrive will wait 15 minutes for the second, after which she will leave. What is the probability that the meeting actually takes place if each of them selects their moment of arrival at random during the interval 12 noon to 1pm?"

The sample space corresponding to the problem is the set of all points (x, y) where $0 \leq x \leq 1$ and $0 \leq y \leq 1$, x& y here represent the fraction of an hour after 12 noon at which each person arrives. So, for example, $(\frac{1}{3}, \frac{1}{5})$ corresponds to person 1 arriving after $\frac{1}{3}$ of an hour, i.e. at 12.20pm and person 2 arriving after $\frac{1}{5}$ of an hour i.e. at 12.12pm. Clearly for these values of x and y the two people will meet.

For them to meet $x - y \leq \frac{1}{4}$ (since 15 minutes $= \frac{1}{4}$ hour)

and $y - x \leq \frac{1}{4}$

Graphing these inequations gives the following shaded region as a representation of the event space. Since the area of the square is 1, we can define the desired probability to be the area of the shaded region.

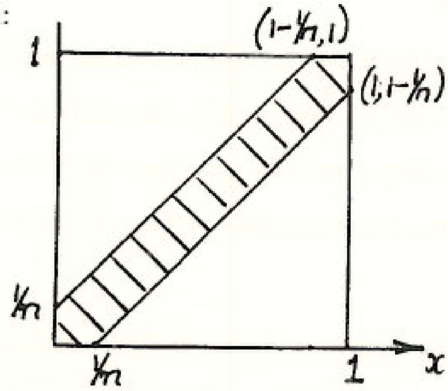


Area

$$= 1 - (\text{Area of} ABE + \text{Area of} DEC)$$
$$= 1 - \frac{9}{16} = \frac{7}{16}$$

One further question of interest is to find how many minutes each person should wait so that they have a 50-50 chance of meeting. i.e. How long should they wait so that the probability is equal to $\frac{1}{2}$?

Let the time be $1/n^{th}$ of an hour. Then proceeding as before, the shaded region is shown, and:



Probability $= \text{Area of shaded region}$

$$= 1 - (1 - 1/n)^2$$

Solving $1 - (1 - 1/n)^2 = \frac{1}{2}$

gives $n = \dfrac{\sqrt{2}}{\sqrt{2} \pm 1}$

The solution $n = \dfrac{\sqrt{2}}{\sqrt{2}+1} = 0.5857$ is meaningless here since clearly $n \geq 1$.

$$\therefore n = \frac{\sqrt{2}}{\sqrt{2}-1} = 3.414214$$

Hence each person should wait $1/n \times 60 \approx 17.6$ minutes to ensure even chances of meeting.

Now try to solve the original problem from the Westpac Competition.