SOLUTIONS OF PROBLEMS 861-871

Q.861 Each of the numbers in a list

$$x_1, x_2, x_3, \cdots, x_n, \cdots$$

is a positive integer written as usual in decimal notation.

For every $n > 1$

$$x_n = x_{n-1} + y_{n-1}$$

where y_{n-1} is the number obtained from x_{n-1} by writing down the digits in reverse order. For example, if $x_n = 100$, then $x_{n+1} = 100 + 001 = 101$, and $x_{n+2} = 101 + 101 = 202$.

Prove that regardless of the value of x_1 from some stage on all the numbers in the list are exactly divisible by 11.

ANS. Everyone knows the test for divisibility by 9: –

$$a_k \times 10^k + a_{k-1} \times 10^{k-1} + \cdots + a_1 \times 10 + a_0$$

is divisible by 9 if and only if

$$a_k + a_{k-1} + \cdots + a_1 + a_0$$

is divisible by 9.

There is a rather similar test for divisibility by 11:– If $a_0, a_1, \cdots, a_k$ are any integers, then $a_k \times 10^k + a_{k-1} \times 10^{k-1} + \cdots + a_1 \times 10 + a_0$ is divisible by 11 if and only if $a_k \times (-1)^k + a_{k-1} \times (-1)^{k-1} + \cdots + a_1 \times (-1) + a_0$ is divisible by 11.

(If you are familiar with congruence notation and properties, the proof of this assertion is very short:– Since $10 \equiv (-1) \pmod{11}$, $10^r \equiv (-1)^r \pmod{11}$ for any positive integer r, and $a_k \times 10^k + \cdots + a_1 \times 10^1 + a_0 \equiv a_k \times (-1)^k + \cdots + a_1 \times (-1) + a_0 \pmod{11}$, from which the result is evident. Otherwise, observe that successive powers of 10 are alternately 1 less and 1 greater than a multiple of 11. With a little labour, the stated result follows from this observation).

One immediate consequence of this test is that if an integer is divisible by 11, so is the integer obtained by writing the digits in reverse order. (Note that

$$a_0 \times (-1)^k + a_1 \times (-1)^{k-1} + \cdots + a_{k-1} \times (-1) + a_k$$
$$= (-1)^k (a_k \times (-1)^k + a_{k-1} \times (-1)^{k-1} + \cdots + a_1 \times (-1) + a_0)).$$

It follows that in the list if one number x_n is divisible by 11, so are y_n and x_{n+1} $(= x_n + y_n)$. So all the following terms are multiples of 11.

Note that if $x_n = a_k \times 10^k + \cdots + a_1 \times 10^1 + a_0$ where k is odd, and the number of digits in x_n is even $(= k + 1)$, then

$$x_n + y_n = (a_k + a_0) \times 10^k + (a_{k-1} + a_1) \times 10^{k-1} + \cdots +$$
$$(a_1 + a_{k-1}) \times 10^1 + (a_0 + a_k).$$

Note that

$$(a_k + a_0) \times (-1)^k + (a_{k-1} + a_1) \times (-1)^{k-1} + \cdots +$$
$$(a_1 + a_{k-1}) \times (-1) + (a_0 + a_k) = 0,$$

since the first term cancels the last, the second cancels the second last, and so on. (Because the number of digits is even, there is no middle term left uncancelled).

By the stated test we see that $x_{n+1}(= x_n + y_n)$ is divisible by 11.

It is obvious that the list $\{x_n\}$ is increasing, and easy to see that the number of digits in x_n is at most one more than the number of digits in x_{n-1}, so that a term x_n with an even number of digits must eventually appear. From the above observations, all later terms in the list are divisible by 11.

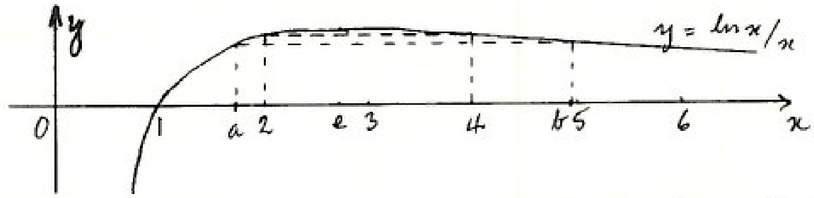
Q.862 **(i)** Sketch the graph of the function $f(x) = \frac{\ell n x}{x} (x > 0)$.
(Here $\ell n x = \log_e x$; $\frac{d}{dx}(\ell n x) = \frac{1}{x}$).
(ii) Show that the only solution of $a^b = b^a$ where a and b are positive integers and $a < b$, is given by $a = 2, b = 4$.

ANS.i)

$$f'(x) = \frac{1}{x}\frac{d}{dx}(\ell n\ x) + \ell n\ x \frac{d}{dx}\left(\frac{1}{x}\right) = \frac{1}{x^2} - \frac{\ell n\ x}{x^2}$$
$$= \frac{1}{x^2}(1 - \ell n\ x).$$

This is positive if $1 > \ell n\ x$, i.e. if $e > x$ and negative if $1 < \ell n\ x$ i.e. if $e < x$.

Hence $f(x)$ has a maximum value when $x = e = 2.71828$, the value there being $f(e) = \frac{\ell n\ e}{e} = \frac{1}{e}$. Note that $\ell n\ x$ is positive if $x > 1$, and negative for $0 < x < 1$. Hence the graph of $f(x)$ must be approximately as shown in the figure.



(ii) If $a^b = b^a$ then, taking logs to base e, $b\ \ell n\ a = a\ \ell n\ b$, $f(a) = \frac{\ell n\ a}{a} = \frac{\ell n\ b}{b} = f(b)$.

If $f(a) = c$, the straight line $y = c$ intersects the graph at two points, P and Q with x-co-ordinates a and b. (Clearly c must be positive since horizontal lines below the x axis intersect the graph of $f(x)$ in only one point). Since the graph has only one stationary point, the maximum at $x = e$, it is clear that a, the x-co-ordinate of P, lies between 1 and $e = 2.71828$, and b the x-co-ordinate of Q is greater than e. (We are assuming $a < b$). Since 2 is the **only** integer between 1 and e, we must have $a = 2$, and then it happens that b has the value 4, also an integer.

Q.863 **(i)** Sketch the graph of the function $g(x) = x\ell nx (x > 0)$.

(ii) For $c > 0$ let $N(c)$ denote the number of solutions of $x^x = c$, $(x > 0)$. Find $N(c)$.

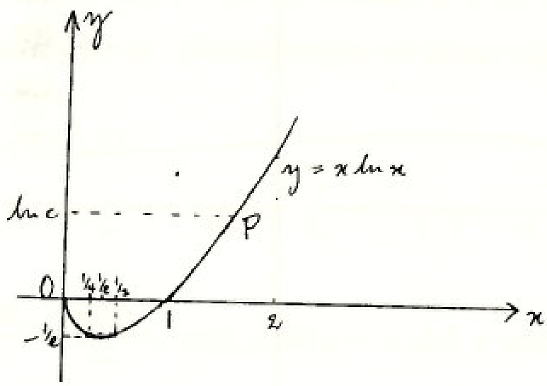
(iii) Find all solutions of $x^x = c$ when $c = \frac{\sqrt{2}}{2}$, when $c = \frac{4\sqrt[3]{36}}{9}$, and when $c = \frac{1}{2}$.

ANS.i) $g'(x) = \frac{dx}{dx}\ell n\ x + x\frac{d}{dx}(\ell n\ x) = \ell nx + 1.$

There is a stationary point when $\ell n\ x = -1$, i.e. when $x = \frac{1}{e}$.

Since $g''(x) = \frac{1}{x}$ which is always positive where $g(x)$ is defined, the graph of $g(x)$ is concave upwards.

Note also that since $\ell n\ x$ is positive for $x > 1$ and negative for $0 < x < 1$, the same is true of $g(x)$. The graph of $g(x)$ must resemble that shown in the sketch. (We shall assume that the value of $g(x)$ tends to 0 as x tends to zero.



This is equivalent to the statement that when y is a large positive number $\frac{\ell n\, y}{y} \approx 0$. (Put $y = \frac{1}{x}$).

Either statement can be proved easily using a result called L'Hôpital's rule which can be found in text books on the calculus, or alternatively from the definition of the ℓn function).

ii) Note that $x^x = c$ if and only if $\ell n\ c = x\ \ell n\ x = g(x)$.

The solutions are the x-co-ordinates of the point(s) P in which the line $y = \ell n\ c$ cuts the graph of $g(x)$. From the graph it is clear that if $\ell n\ c \geq 0$ (i.e. if $c \geq 1$ there is only one point P, so $N(c) = 1$ if $c \geq 1$. If $\ell n\ c < -\frac{1}{e}$ there is no point of intersection, i.e. $N(c) = 0$ if $0 < c < e^{-\frac{1}{e}}$.

If $e^{-\frac{1}{e}} < c < 1$, i.e. if $-\frac{1}{e} < \ell n\ c < 0$, the horizontal line will meet the graph of $g(x)$ in two points, and $N(c) = 2$. Finally $N(e^{-\frac{1}{e}}) = 1$, since $y = -\frac{1}{e}$ is tangential to the graph.

iii) Since $e^{-\frac{1}{e}} \approx .6922$, $\frac{\sqrt{2}}{2}$ lies in the range $(e^{-\frac{1}{e}}, 1)$, so $N(\frac{\sqrt{2}}{2}) = 2$. In fact $\frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}} = (\frac{1}{2})^{\frac{1}{2}} = (\frac{1}{4})^{\frac{1}{4}}$, so the values of x satisfying $x^x = \frac{\sqrt{2}}{2}$ are $x = \frac{1}{2}$, $x = \frac{1}{4}$. Since $0 < \frac{1}{2} < e^{-\frac{1}{e}}, N(\frac{1}{2}) = 0$.

When $c = \frac{4}{9}\sqrt[3]{36}(> 1)$, $N(c) = 1$.

Since $\frac{4}{9}\sqrt[3]{36} = (\frac{4}{3})^{\frac{4}{3}}$, $x = \frac{4}{3}$ is the only solution.

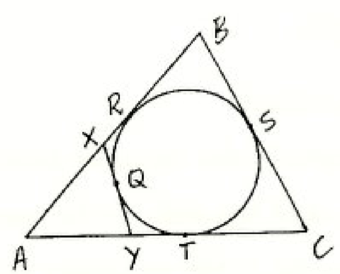
Q.864 In a club with 36 members any two members are either friends or enemies, and each member has exactly 13 enemies. In how many different ways can one select three members so that they are either all friends or all enemies.

ANS. Represent the club members by 36 points in space. Join every pair of points by a line segment, coloured red if the corresponding members are enemies, but coloured green if they are friends. The question can now be restated "How many different one-colour triangles (either red or green) have been formed?"

Let A be any one of the points. Denote by X the set of points (13 in number) joined to A by a red line segment, and by Y the set of 22 points joined to A by a green line. Of the ${}^{13}C_2$ line segments with both end points in X, let r denote the number which are red. Similarly let g denote the number of green line segments with both end points in Y. Then clearly the number of one-colour triangles having A as a vertex is equal to $r+g$.

Each of the 13 points in X is the end point of 13 different red lines. Of these 169 red "end of lines", 13 have come from A, and $2 \times r$ have come from the r red lines with both ends in X, each of which has been counted twice. That leaves $(13^2 - 13 - 2r)$ which have their other end in Y. Similarly there are $22^2 - 22 - 2g$ green lines starting from Y which have their other end point in X. But since each of the 13×22 lines joining a point in X to a point in Y is either red or green, we deduce that $13 \times 22 = (13^2 - 13 - 2r) + (22^2 - 22 - 2g)$. Hence $r + g = 166$. Since every one of the 36 points is a vertex of 166 one-colour triangles, the total number of one-colour triangles is $(166 \times 36) \div 3$. [Division by 3 is necessary, since each triangle has been counted three times, once for each vertex.] Hence the final result is 1992.

Q.865 For any triangle $\triangle ABC$ let X, Y be points in the sides AB, AC respectively such that $XY \| BC$ and XY is tangential to the inscribed circle of the triangle. (See figure).



Prove that the length XY cannot exceed $\frac{1}{8}$th of the perimeter of $\triangle ABC$. Is equality possible?

ANS. We shall use later the result that for any positive numbers a, b we have $\dfrac{ab}{(a+b)^2} \leq \dfrac{1}{4}$, with equality when $a = b$.

This follows from $4ab \leq 4ab + (a-b)^2 = (a+b)^2$.

In the figure, Q, R, S, T are the points of contact of the tangents. We have $AR = AT$, $CS = CT$, $BR = BS$, $XQ = XR$ and $YQ = YT$. If p_1 denotes the perimeter of $\triangle AXY$, and p_2 that of $\triangle ABC$, then

$$\begin{aligned} p_1 + 2BC &= AX + (XQ + QY) + AY + 2BS + 2CS \\ &= (AX + XR) + (AY + YT) + RB + BS + SC + TC \\ &= (AR + RB) + (AT + TC) + (BS + SC) = p_2. \end{aligned}$$

Since $\triangle AXY$ is similar to $\triangle ABC$.

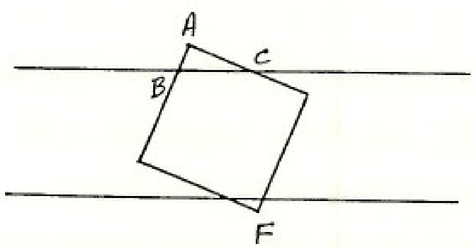
$$\frac{XY}{BC} = \frac{p_1}{p_2}, \text{ so that } \frac{XY}{p_2} = \frac{BC.p_1}{p_2^2} = \frac{1}{2}\frac{(2BC)p_1}{p_2^2}$$

Using the preliminary result, with $p_1 + 2BC = p_2$, we have $\dfrac{2BC.p_1}{p_2^2} \leq \dfrac{1}{4}$ with equality when $p_1 = 2BC = \frac{1}{2}p_2$.

Hence $\dfrac{XY}{p_2} \leq \dfrac{1}{8}$ with equality when $2BC = \frac{1}{2}p_2$.

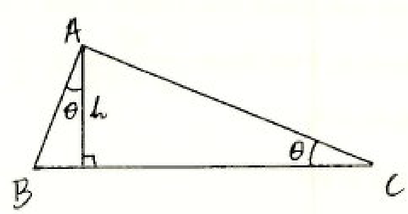
i.e. Equality is attained when the side BC of the triangle is one quarter of the perimeter.

Q.866 A 1 metre square masonry slab which formed part of a 1 metre wide path has become displaced as shown in the figure.



A workman repairing the path saws off the two triangular pieces which project beyond the sides of the path. Find the sum of the perimeters of the two triangles.

ANS. Let θ be the angle through which the slab has turned.



If h is the perpendicular distance from the projecting corner A of the slab to the nearest edge of the path (see fig.1) the perimeter of the overhanging triangle is

$$AB + AC + BC = h\left(\frac{1}{\cos\theta} + \frac{1}{\sin\theta} + \frac{1}{\sin\theta\cos\theta}\right).$$

Similarly the perimeter of the overhanging triangle on the other side is $k\dfrac{(\sin\theta + \cos\theta + 1)}{\sin\theta\cos\theta}$ where

k is the perpendicular distance from F to the edge of the path.

The sum of these perimeters is $(h+k)\dfrac{(\sin\theta+\cos\theta+1)}{\sin\theta\cos\theta}$ (1)

Now the diagonal AF of the slab has length $\sqrt{2}$ metres and is inclined at the angle $\angle AFM = (45°+\theta)$ to the direction of the path. The side AM in the right angled triangle in figure 2 is

$$\begin{aligned} AM &= \sqrt{2}\sin(45°+\theta) = \sqrt{2}(\sin 45° \cos\theta + \cos 45° \sin\theta) \\ &= \sin\theta + \cos\theta \end{aligned}$$

Hence, subtracting the width of the path

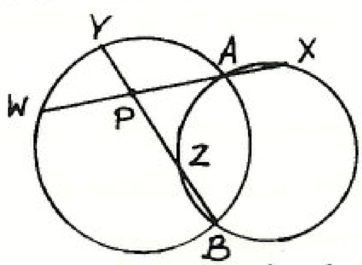
$$h + k = \sin\theta + \cos\theta - 1$$

Substituting this into (1) gives

$$\begin{aligned} \text{sum of perimeters } &= \frac{((\sin\theta+\cos\theta)-1)((\sin\theta+\cos\theta)+1)}{\sin\theta\cos\theta} \\ &= \frac{(\sin\theta+\cos\theta)^2-1}{\sin\theta\cos\theta} \\ &= \frac{\sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta - 1}{\sin\theta\cos\theta} \\ &= 2. \end{aligned}$$

Thus (provided there is overhang on both sides of the path) the two perimeters add to 2 metres.

Q.867 Two circles intersect in A and B. P is a point inside one of the circles.



AP (produced) cuts the first circle in W, the second in X. BP (produced) cuts the first circle in Y, the second in Z. If $WYXZ$ is a rhombus, prove that the two circles are of equal radius.

ANS. Since the diagonals of a rhombus bisect each other at right angles, if $WYXZ$ is a rhombus B lies on the perpendicular bisector of WX. Hence $\triangle BWP$ is congruent to $\triangle BXP$ and we deduce that $\angle BWA = \angle BXA$. Since the common chord AB subtends equal angles at the circumferences of the two circles, they are of equal radius.

Q.868 Twelve sentry posts are situated (not necessarily equally spaced) on the circular wall of a citadel. At 12 noon, a sentry leaves each post in one direction or the other, marching at a speed which would make a complete circuit in exactly one hour. If two sentries meet, they both about turn and continue marching at the same speed in opposite directions.

Prove that at midnight each sentry will be exactly at his own starting post.

In fact prove that this must already be the case at 6pm.

ANS. An observer seeing two sentries just before their meeting and again just after (but not at the critical moment) would not be able to determine if they had done their "about turn" manoeuvre, or simply marched past each other, unless he was close enough to recognize them as individuals. Hence at any instant the disposition of marching (anonymous) sentries is exactly the same as if all "about turns" were replaced by marching past. Hence after one hour there is a sentry at each of the twelve posts, marching in the direction taken initially by the original occupant. (*)

Now, since in fact no sentry ever passes another, the order of the twelve sentries aroung the wall is exactly the same as at the starting time. Hence if after the first hour one sentry finds himself at the kth post clockwise from his starting point, all twelve of the sentries must have moved k posts clockwise by 1pm.

Of course because of (*), after the next hour each sentry will have moved another k posts clockwise (and will now be marching in the direction taken by the original occupant of the new post.)

In fact, after n hours each sentry will have moved nk posts clockwise. So in twelve hours, each sentry will have moved $12k$ posts clockwise, i.e. k complete circuits clockwise, and will be exactly at his own starting post.

But what about 6pm?

If in an hour the sentries have moved k posts clockwise, then the total distance marched by all sentries in the clockwise direction is k circuits more than the distance marched in the anticlockwise direction. Suppose initially m sentries moved clockwise and the other $(12 - m)$ marched anticlockwise. Then the same applies at any other instant, and in one hour the total distances marched are m circuits clockwise, and $(12-m)$ anticlockwise. Hence $k = m-(12-m) = 2(m-6)$. Thus k is an even number, so the number of posts shifted clockwise in 6 hours, viz $6k$, is already a multiple of 12. Thus at 6pm each sentry has moved a whole number of circuits and is therefore back at his starting post.

(In the above, clockwise may be replaced everywhere by anticlockwise, or alternatively k may be permitted to be a negative integer if the net movement of each sentry in an hour is in the anticlockwise direction.)

Q.869 Let c_n be the nth term of the sequence defined by $c_1 = 1,\ c_2 = -1,$ $c_n = -c_{n-1} - 2c_{n-2}$ for $n \geq 3$.

Prove that $2^{n+1} - 7c_{n-1}^2$ is a perfect square for every integer $n \geq 2$.

ANS. First, some exploration. Calculate the rows in the following table (in which $d_n^2 = 2^{n+1} - 7c_{n-1}^2$).

n	1	2	3	4	5	6	7	8	9	10	11	$\cdots$
c_n	1	-1	-1	3	-1	-5	7	3	-17	11	23	$\cdots$
$(2^{n+1} - 7c_{n-1}^2)$	–	1	9	25	1	121	81	169	961	25	3249	$\cdots$
d_n	–	-1	-3	5	1	-11	9	13	-31	5	57	$\cdots$

In calculating d_n, we have chosen the signs as shown because we have noticed that by so doing the sequence $\{d_n\}$ obeys the same recursion rule as $\{c_n\}$ i.e.

$$d_n = -d_{n-1} - 2d_{n-2}. \tag{1}$$

This may lead us to look for a relation between d_n and the terms in the sequence $\{c_n\}$, and if we are lucky enough we might notice that $d_n = 2c_n + c_{n-1}$ for $n = 2, \cdots, 11$. (*)

If so, we can attempt to prove by mathematical induction that $2^{n+1} - 7c_{n-1}^2 = (2c_n + c_{n-1})^2$ for $n \geq 2$. This is true for $n = 2$, obviously. If it is true for $n = k$

then

$$\begin{aligned}(2c_{k+1}+c_k)^2 &= (2(-c_k-2c_{k-1})+c_k)^2 = (-c_k-4c_{k-1})^2\\ &= c_k^2+8c_kc_{k-1}+16c_{k-1}^2\\ &= 14c_{k-1}^2-7c_k^2+2(4c_k^2+4c_kc_{k-1}+c_{k-1}^2)\\ &= 14c_{k-1}^2-7c_k^2+2(2c_k+c_{k-1})^2\\ &= 14c_{k-1}^2-7c_k^2+2(2^{k+1}-7c_{k-1}^2)\\ &= 2^{k+2}-7c_k^2.\end{aligned}$$

This completes the induction step (i.e. it is true for $n=k+1$ whenever it is true for $n=k$) so the desired result is true for all $n \geq 2$.

If we are not lucky enough to notice (*), we can still succeed, proving by induction the proposition:-

"$d_n^2 = 2^{n+1}-7c_{n-1}^2$ and $2^{n+1}+4d_nd_{n-1} = -28c_{n-1}c_{n-2}$" for $n \geq 3$, the first part of which is the desired result.

(Here d_n is defined by $d_2=-1, d_3=-3, d_n=-d_{n-1}-2d_{n-2}$). We omit the details.

Q.870 For integers $1 \leq k \leq 1992$ denote by s_k the sum

$$\frac{1}{k}+\frac{1}{k+1}+\frac{1}{k+2}+\cdots+\frac{1}{1992}.$$

Determine $s_1+s_1^2+s_2^2+s_3^2+\cdots+s_{1992}^2$.

ANS. When $(a_1+a_2+\cdots+a_n)$ is squared, the result is $a_1^2+\cdots+a_n^2+2a_1a_2+2a_1a_3+\cdots+2a_1a_n+2a_2a_3+\cdots+2a_2a_n+\cdots+\cdots+2a_{n-1}a_n = \sum_{j=1}^{n} a_j^2+2\sum_{1\leq k<\ell\leq n} a_ka_\ell$.

If all of $s_1^2, s_2^2, \cdots, s_{1992}^2$ are expanded in this way the resulting terms in $s_1+s_1^2+s_2^2+s_k^2+\cdots+s_{1992}^2$ can be regrouped as

$$T_1+T_2+\cdots+T_t+\cdots+T_{1992}$$

where $T_t = \frac{1}{t}+\sum_{j=1}^{t} t^2+2\sum_{k=1}^{t-1}\left(\sum_{r=k}^{t-1}\frac{1}{rt}\right)$.

For example,

$T_4 = \frac{1}{4}$ (appearing in s_1) $+(\frac{1}{4^2} + \frac{1}{4^2} + \frac{1}{4^2} + \frac{1}{4^2})$ (appearing in $s_1^2, s_2^2, s_3^2, s_4^2$) + $2[(\frac{1}{1} + \frac{1}{2} + \frac{1}{3})\frac{1}{4} + (\frac{1}{2} + \frac{1}{3})\frac{1}{4} + \frac{1}{3}\frac{1}{4}]$ (product terms $\frac{2}{rt}$ with $t = 4$ and $r < t$ appearing in s_1^2, s_2^2, and s_3^2.)

Note that $T_4 = \frac{1}{4} + 4 \times \frac{1}{4^2} + 2 \times \frac{1}{4}[\frac{1}{1} + 2 \times \frac{1}{2} + 3 \times \frac{1}{3}] = 2.$

In fact, for any t

$$T_t = \frac{1}{t} + t \times \frac{1}{t^2} + \frac{2}{t}\left[\left(\frac{1}{1} + \cdots + \frac{1}{(t-1)}\right) + \left(\frac{1}{2} + \cdots + \frac{1}{t-1}\right) + \cdots + \frac{1}{t-1}\right]$$

$$= \frac{1}{t} + \frac{1}{t} + \frac{2}{t}\left[\frac{1}{1} + 2 \times \frac{1}{2} + 3 \times \frac{1}{3} + \cdots + (t-1) \times \frac{1}{t-1}\right] = \frac{1}{t} + \frac{1}{t} + \frac{2}{t}(t-1) = 2.$$

Therefore the required sum is $T_1 + T_2 + \cdots + T_{1992}$

$= 2 + 2 + \cdots + 2 = 2 \times 1992 = 3984.$

Q.871 The polynomial $p(x)$ has degree n and

$$p(k) = 3^k \text{ for } k = 0, 1, 2, \cdots, n.$$

Find $p(n+1)$.

ANS.

$p_0 = 3^0 = 1 \Rightarrow p(x) = 1 + xp_1(x)$ where $p_1(x)$ is a polynomial of degree $(n-1)$

$$p(1) = 3 \Rightarrow 1 + 1p_1(1) = 3 \Rightarrow p_1(1) = 2 \Rightarrow p_1(x) = 2 + (x-1)p_2(x)$$

$$\Rightarrow p(x) = 1 + 2x + x(x-1)p_2(x) \text{ where } p_2(x) \text{ is of degree } (n-2).$$

$$p(2) = 3^2 \Rightarrow 1 + 2 \times 3 + 2!\ p_2(2) = 9 \Rightarrow 2!p_2(2) = 4 = 2^2$$

$$\Rightarrow p_2(x) = \frac{2^2}{2!} + (x-2)p_3(x)$$

$$\Rightarrow p(x) : 1 + \frac{2}{1!}x + \frac{2^2}{2!}x(x-1) + x(x-1)(x-2)p_3(x)$$

where $p_3(x)$ has degree $(n-3)$.

$$p(3) = 3^3 \Rightarrow 1 + 2 \times 3 + \frac{2^2}{2} \times 3 \times 2 + 3!p_3(3) = 3^2 \Rightarrow p_3(3) = \frac{2^3}{3!}$$

$$\Rightarrow p_3(x) = \frac{2^3}{3!} + (x-3)p_4(x) \Rightarrow (x-2)$$

$$\Rightarrow p(x) = 1 + \frac{2x}{1!} + \frac{2^2}{2!}x(x-1) +$$

$$+ \frac{2^3}{3!}x(x-1)(x-2) + x(x-1)(x-2)(x-3)p_4(x)$$

where $p_4(x)$ has degree $(n-4)$. Recognising the pattern we consider the polynomial of degree n defined by

$$P(x) = 1+\frac{x}{1!}2+\frac{x(x-1)}{2!}2^2+\cdots+\frac{x(x-1)\cdots(x-r+1)}{r!}2^r+\cdots+\frac{x(x-1)\cdots(x-n+1)}{n!}2^n$$

If k is any integer $\leq n$.

$$P(k) = 1+\frac{k}{1!}2+\frac{k(k-1)}{2!}2^2+\cdots+\frac{k(k-1)\cdots 2}{k!}2^k+0+\cdots+0.$$

(the zeros resulting from the factor $(x-k)$ in any terms for $r > k$).

$$\begin{aligned} P(k) &= 1+{}^k C_1 2+{}^k C_2 2^2+\cdots+{}^k C_k 2^k \\ &= (1+2)^k \text{ by the Binomial Theorem} \\ &= 3^k \end{aligned}$$

Hence $P(x)$ is indeed indentical with $p(x)$.

$$\begin{aligned} \therefore p(n+1) &= 1+\frac{(n+1)}{1!}2+\frac{(n+1)n}{2!}2^2+\cdots\frac{(n+1)\cdots 2}{n!}2^n \\ &= 1+{}^{n+1}C_1 2+{}^{n+1}C_2 2^2\cdots+{}^{n+1}C_n 2^n = (1+2)^{n+1}-2^{n+1} \\ &= 3^{n+1}-2^{n+1}. \end{aligned}$$