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SUM FUN WITH SERIES

Peter Brown

You have (hopefully) learnt a little bit at school about geometric series, and have studied so-called 'limiting sums' of geometric series.

For example $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots=2$, which means that $1+\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^n}$ gets 'close to' 2 as n gets large.

In general, the infinite geometric series with first term a, and common ratio r has the sum,

$$a+ar+ar^2+\cdots=\frac{a}{1-r} \text{ if and only if } -1<r<1.$$

I want firstly to show you how to play with some simple geometric series to derive some rather lovely results about some non-geometric series. For example, can we sum

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\cdots ?$$

Two problems arise:

(1) Does a given series actually get 'close to' some finite number if we take lots of terms?

(2) How do we find that number if it exists?

For example, the series
$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots$ continues to get larger the more terms we take,

e.g.,
$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{10^6}\simeq 14.39$$

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{10^{100}}\simeq 230.84.$$

In fact it gets "arbitrarily large", larger than any positive number we choose and we say that the series **diverges** to infinity. You might be able to show that the sum of the first n terms lies between $\ell n n$ and $\ell n n+1$ by considering the graph of $y=\frac{1}{x}$. (Indeed $\ell n n+\frac{1}{2}+\frac{1}{12}$ is generally a very good approximation.) However we can quickly show it diverges to infinity by grouping in powers of 2 as follows:

$$
\begin{aligned}
1+\frac{1}{2}+(\frac{1}{3}+\frac{1}{4})+(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8})+(\frac{1}{9}+\cdots+\frac{1}{16})+\cdots \\
>1+\frac{1}{2}+2.\frac{1}{4}+4.\frac{1}{8}+8.\frac{1}{16}+\cdots \\
=1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots.
\end{aligned}
$$

However, the series

$$
\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots \simeq 1.644933 \text{ taking } 10^6 \text{ terms}
$$

$$
\text{and is } \simeq 1.644934 \text{ taking } 10^{100} \text{ terms.}
$$

This series in fact **converges** to $\frac{\pi^2}{6}(\simeq 1.6449345)$. This was proved by the great Swiss mathematician Leonard Euler in about 1736.

There are numerous tests to see whether or not a series converges, but I don't want to discuss them here. You will have to take my word for it that the series following do in fact converge, but you can check numerically on your home computer.

The trouble with numerical checks is that they don't always point you in the right direction. For example, the series

$$
\frac{1}{2\ell n2}+\frac{1}{3\ell n3}+\frac{1}{4\ell n4}+\cdots
$$

diverges to infinity but does so very slowly. This can be seen if it is accepted that the sum of the first n term is approximately $\ell n(\ell n\ n)$. The sum of the first 10^6 terms is 2.62579. Whilst the sum of the first 10^{100} terms is still just $5.4392\cdots$. If you remember that 10^{100} is about the number of atoms in the universe you might sense that the series is convergent and guess a limit well under 5.

The second question above is (for me at least) the more interesting. Infinite series are, in general, fairly 'dangerous' objects, and some of the tricks I'm about to show you don't always work, but again you'll have to take my word for it that they do in the examples I'll be using.

Suppose $-1<x<1$, then the following geometric series has $a=1$ and $r=x$, so

$$
1+x+x^2+x^3+\cdots=\frac{1}{1-x}. \tag{1}
$$

Putting $x = \frac{1}{2}$, gives the well-known result, (mentioned above)

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2.$$

Differentiating both sides of (1) gives

$$1 + 2x + 3x^2 + 4x^3 + \cdots + nx^{n-1} + \cdots = \frac{1}{(1-x)^2} \text{ (Check this!)}$$

Putting $x = \frac{1}{2}$ (say) gives

$$1 + \frac{2}{2} + \frac{3}{2^2} + \frac{4}{2^3} + \cdots + \frac{n}{2^{n-1}} + \cdots = 4.$$

Note that this is no longer a geometric series.

Differentiating both sides of (1) twice, gives

$$2 + 6x + 12x^2 + 20x^3 + \cdots + n(n-1)x^{n-2} + \cdots = \frac{2}{(1-x)^3}$$

Substituting, say $x = \frac{1}{3}$, gives

$$2 + \frac{6}{3} + \frac{12}{3^2} + \frac{20}{3^3} + \cdots + \frac{n(n-1)}{3^{n-2}} + \cdots = \frac{27}{4}.$$

You can choose any value of x (between -1 and 1) and get new series. I invite you to experiment with this and perhaps check on your home computer.

If we integrate both sides of (1), we have

$$-\ell n(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^{n+1}}{n+1} + \cdots + C. \tag{2}$$

Now if $x = 0$, we find that $C = 0$.

Putting $x = \frac{1}{2}$, we have

$$\ell n2 = \frac{1}{2} + \frac{1}{2.2^2} + \frac{1}{3.2^3} + \frac{1}{4.2^4} + \cdots + \frac{1}{(n+1).2^{n+1}} + \cdots$$

Series (2) in fact does converge if $x = -1$ (even though series (1) does not!), so putting $x = -1$, we have

$$\ell n2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \cdots$$

which is a rather lovely result. A little experimentation will show that this is not a good way to approximate $\ell n2$.

Next consider the series

$$1 - x^2 + x^4 - x^6 + x^8 - x^{10} + \cdots = \frac{1}{1+x^2} \text{ (since } a = 1,\ r = -x^2)$$

which converges for $-1 < x < 1$.

Integrate both sides to get

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \frac{x^{11}}{11} + \cdots + C.$$

Again, if we put $x = 0$, we find that $C = 0$.

This series also converges for $x = 1$, giving

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots$$

This is, I hope you agree, a very pretty and perhaps surprising result, relating π and the odd numbers.

Taylor - Maclaurin Series: It can be shown (ask your teacher!), that under certain conditions, a function $f(x)$ can be written as an infinite series,

$$f(x) = f(0) + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

where $f^n(0)$ means 'differentiate $f(x)$ n times and substitute $x = 0$".

For example, if $f(x) = e^x$, then $f'(0) = f''(0) = \cdots = f^{(n)}(0) = 1$, so

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

It turns out that this series converges for **all** values of x, so if $x = 1$ we have

$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \cdots = e.$$

Similarly, you can show that

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \tag{3}$$

Differentiating gives:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad \text{(Check these two formulae)}$$

Note that both these series converge for any value of x you like.

An Application: As you may be aware, there are many integrals which, although they look harmless enough, cannot be evaluated in terms of 'elementary functions'. The best known example is perhaps $\int e^{x^2}dx$. Such integrals can, however, be approximated by using series. For example, the integral

$$\int \sin(x^2)dx$$

turns up in physics (in the theory of optics to be precise) and cannot be explicitly evaluated in terms of the functions we know and love. (It is called a Fresnel integral).

If we wanted to get an approximate value of $\int_0^1 \sin(x^2)dx$ we can start with series (3),

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

Replacing x with x^2 and integrating, we get

$$\begin{aligned}
\int_0^1 \sin(x^2)dx &= \int_0^1 x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots dx \\
&= \frac{1}{3} - \frac{1}{7 \times 3!} + \frac{1}{11 \times 5!} - \cdots \\
&\simeq \frac{1}{3} - \frac{1}{7 \times 3!} \text{ to 3 decimal places (since } \frac{1}{11 \times 5!} \text{ is small)} \\
&\simeq 0.310.
\end{aligned}$$

Whether your mathematical interests lie in finding beautiful formulae like

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

or in practical applications, I think you'll find that series are both fun and useful.

In the last issue we posed the following question.

"Two companies, A and B, offer starting salaries of \$20000 but A gives an annual rise of \$2000 whilst B gives a half-yearly rise of \$500. Which company should we work for if we wish to maximize our income?"

The question is discussed in Morris Kline's book **Mathematics and the Search for Knowledge** (OUP 1986). He claims that most people would choose A even when it is understood that B is to earn \$10500 in the 2nd half of his first year. There are other interpretations of how B's income grows (see Rod James' article) but the question still supports Kline's assertion that our intuition is not always reliable. The following note by George Harvey solves the general question with the interpretation of Kline.

SALARY OPTIONS

George Harvey*

(1) Under what condition is an increment of \$$b$ paid k times a year [Option (B)] more advantageous than an annual increment of \$$a$ paid yearly [Option (A)]?

(2) When the condition in (1) is satisfied, what is the least number of years before Option (B) establishes its superiority?

[Assume that increments are effective immediately. It is obvious that the merits of the options are independent of current salary which we may therefore assume with loss of generality to be zero].

Notation: A_n, B_n = salary in nth year under option $(A), (B)$ respectively.

(1) **Salary in nth year** $\quad A_n = na$

$$B_1 = (1 + 2 + 3 + \cdots + k)b$$

$$B_2 = [(k+1) + (k+2) + (k+3) + \cdots + (k+k)]b$$

* George is a teacher at St. Clare's College, Griffith, A.C.T.