SALARIES, LOANS AND DIFFERENCE EQUATIONS

Rod James

In the last issue of **Parabola**, the question was posed as to whether it is better to work for Company A (with an annual rise of \$2,000) or Company B (with only a six-monthly rise of \$500). In an article in this issue, George Harvey shows, for example, that A's salary is \$500 per year better if we interpret the question to mean that

(a) B's half-yearly rise is \$500 per half-year (and not \$500 per year); and

(b) the increments are retrospective (i.e. in the first year A's salary is \$20,000 + \$2,000 rise; whilst B's salary is \$20,000 + \$500 rise in the first 6 months + \$500 + \$500 rise in the second 6 months, etc.).

However, these interpretations are not necessarily the usual ones in practice. We will look at them in the reverse order to the above.

1) In business "an annual rise of \$2,000" usually means that the salary for the first year is \$20,000, the salary for the second year is \$22,000 etc. Applying the same interpretation to company B, the salary for the first 6 months is $\$20{,}000 \times \frac{1}{2} = \$10{,}000$, the salary for the second 6 months is \$10,500 (and so the salary for the first year is \$20,500) etc. Continuing this, we have

Year	A's salary	B's salary
1	\$20,000	\$20,500
2	\$22,000	\$22,500 (= \$11,000 + \$11,500)
3	\$24,000	\$24,500

etc.

This shows that B's salary is actually \$500 per year better than A's salary.

2) Again, in practice "a ... rise of \$500" usually means a rise of \$500 in the annual salary, i.e., a rise of \$250 in a 6-monthly period. Using this, the above table becomes

Year	A's salary	B's salary
1	\$20,000	\$20,250
2	\$22,000	\$21,250 (= \$10,500 + \$10,750)
3	\$24,000	\$22,250 (= \$11,000 + \$11,250)

etc.

Under this interpretation, it is clear that (apart from the first year) B's salary is less than that of A. In fact, using the same reasoning as George, it can be shown that in the nth year,

$$\text{A's salary - B's salary} = \$(1000n - 1250)^*$$

We turn now from the earning of a salary to the repayment of a loan, where the mathematics is a little more complicated. Suppose you were to borrow $20,000 at an interest rate of a mere 1% per month (better than most credit cards!) and to repay it at $200 per month. Before reading on you might like to try to guess (or estimate) how long it will take to repay this loan.

As usual in Mathematics we look in preference at the general situation of borrowing $\$A$ at a monthly interest rate of $(100r)\%$ with a monthly repayment of $\$R$. We write $\$A_n$ for the amount still owing after n months (and so $A_0 = A$). In the $(n+1)$th month, the amount owing $\$A_n$ increases by the interest $\$(rA_n)$ on it and decreases by the repayment of $\$R$. So at the end of $n+1$ months the amount (in dollars) still owing is

$$A_{n+1} = A_n + rA_n - R$$

$$A_{n+1} = (1+r)A_n - R. \tag{1}$$

Our aim is to find an explicit formula for A_n which we do in several steps.

1. What happens if no repayments are made at all (i.e. $R = 0$)?

   In this case the interest owing compounds, and equation (1) becomes

   $$A_{n+1} = (1+r)A_n. \tag{$*$}$$

   You may also recognise this as a geometric sequence with common ratio $1+r$. Using the formula for compound interest (or for the n'th term of a geometric sequence), the formula given by (*) is

   $$A_n = A(1+r)^n.$$

   (Note: It is not $A(1+r)^{n-1}$ as the first term A_1 is $A(1+r)$).

---

*Moral: Always find out what the other person really means!

2. Now imagine a situation where the repayments exactly meet the interest and repay none of the capital. In this case, the amount A_{n+1} owing at the end of $n+1$ months will be the same as the amount A_n. Substituting $A_{n+1} = A_n$ into (1) gives

$$A_n = (1+r)A_n - R$$
$$A_n = R/r \qquad (**)$$

So if an amount of \$ (R/r) is owing at the end of n months, then

$$\begin{aligned} A_{n+1} &= (1+r)A_n - R \\ &= (1+r)R/r - R = R/r \end{aligned}$$

and so an amount of \$$(R/r)$ dollars is still owing at the end of $n+1$ months.

We now return to the original problem of "solving" the equation (1), i.e. finding an explicit formula for A_n. First note that (as we have seen) the formula $(**)$ satisfies (1). To see how far another solution can be form this, we subtract it:

$$\begin{aligned} B_n &= A_n - R/r \\ B_{n+1} &= A_{n+1} - R/r = (1+r)A_n - R - R/r \quad \text{(from (1))} \\ &= (1+r)(B_n + R/r) - R - R/r \\ B_{n+1} &= (1+r)B_n \end{aligned}$$

This is the geometric series again, and so we know that

$$B_n = B_0(1+r)^n$$

where $B_0 = A_0 - R/r = A - R/r$. So the solution to (1) is

$$\begin{aligned} A_n &= B_n + R/r \\ &= B_0(1+r)^n + R/r \\ A_n &= (A - R/r)(1+r)^n + R/r. \end{aligned}$$

In particular if you borrow \$20,000 at 1% per month and repay it at \$200 per month, then $A = 20,000, r = 1/100, R = 200$. So

$$A_n = (20,000 - 20,000) \times 1.01^n + 20,000 = 20,000$$

and so you will be repaying the loan for the rest of your life! You might now like to answer a similar problem given to me by my brother:

What is the monthly repayment on a 20 year loan of \$100,000 at an annual interest rate of $7\frac{1}{2}$% (calculated monthly)?

As well as being useful in everyday life, the above is an example of an interesting branch of mathematics called difference equations. This consists in finding an explicit formula for an expression x_n when x_{n+1} is given as an expression in the previous terms). Some examples are:

1. Geometric sequence: $x_{n+1} = rx_n$
2. Arithmetic sequence: $x_{n+1} = x_n + d$
3. Fibonacci sequence: $F_{n+2} = F_{n+1} + F_n$ (See page 11 of **Parabola** Vol.28, No.2).

The idea of the geometric sequence can also be applied to Fibonnaci sequences. The Fibonnaci numbers are numbers F_n where $F_0 = F_1 = 1$ and, for $n > 1$,

$$F_{n+2} = F_n + F_{n+1}. \tag{2}$$

Although this is not the usual equation (like (1)) for a geometric sequence, its solution might still be a geometric sequence. In trying to find the common ratio r, suppose that $F_n = Ar^n$. Then $F_{n+1} = Ar^{n+1}$ and

$$\begin{aligned} Ar^{n+2} &= F_{n+2} \\ &= F_{n+1} + F_n = Ar^{n+1} + Ar^n. \end{aligned}$$

Dividing by Ar^n, this becomes the quadratic equation

$$\begin{aligned} r^2 &= r + 1 \\ r &= \frac{1 \pm \sqrt{5}}{2} \end{aligned}$$

So (**) has the two solutions $F_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n$ and $F_n = B\left(\frac{1-\sqrt{5}}{2}\right)^n$. Now it is not hard to show that

If F'_n and F''_n are two solutions of (2) then $F'_n + F''_n$ is also a solution of (2).

This means that (2) has solutions of the form

$$F_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

Substituting $n = 0$ and 1, we get

$$1 = F_0 = A\left(\frac{1+\sqrt{5}}{2}\right)^0 + B\left(\frac{1-\sqrt{5}}{2}\right)^0 = A + B$$

$$1 = F_1 = A\left(\frac{1+\sqrt{5}}{2}\right)^1 + B\left(\frac{1-\sqrt{5}}{2}\right)^1 = (A(1+\sqrt{5}) + B(1-\sqrt{5}))/2.$$

Substituting the first equation into the second,

$$(A - B)\sqrt{5} = 1$$

$$A + B = 1$$

so $A = (\sqrt{5}+1)/2\sqrt{5},\ B = (\sqrt{5}-1)/2\sqrt{5}$ and the formula for F_n becomes

$$\begin{aligned} F_n &= \frac{\sqrt{5}+1}{2\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{\sqrt{5}-1}{2\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n \\ &= \left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right)/2\sqrt{5}. \end{aligned}$$

It is remarkable that this expression is a positive integer for every positive integer n (you may care to check this).

Let us try this approach on another difference equation

$$x_{n+2} = 4x_{n+1} - 4x_n \tag{3}$$

where $x_0 = 1$ and $x_1 = 4$.

Writing $x_n = Ar^n$, we get (as before) the quadratic equation

$$r^2 = 4r - 4$$

$$r = 2.$$

However this time there appears to be only one answer to the quadratic and the solution appears to be $x_n = A2^n$. But it is not hard to see that $x_n = n2^n$ is also a solution of (3), and so A must depend on n. Writing $x_n = A_n 2^n$, equation (3) yields

$$\begin{aligned} 2^{n+2} A_{n+2} &= x_{n+2} \\ &= 4x_{n+1} - 4x_n \\ &= 4 \times 2^{n+1} A_{n+1} - 4 \times 2^n A_n. \end{aligned}$$

Dividing by 2^{n+2}, this becomes

$$A_{n+2} = 2A_{n+1} - A_n.$$

Writing $B_n = A_{n+1} - A_n$ (and so $B_{n+1} = A_{n+2} - A_{n+1}$) we get

$$B_{n+1} = A_{n+2} - A_{n+1} = A_{n+1} - A_n = B_n$$

so $B_n = B_{n-1} = \cdots = B_0$.

If we write d for B_0, then

$$A_{n+1} - A_n = B_n = d$$

$$A_{n+1} = A_n + d$$

which is an Arithmetic Sequence! Using the formula for an Arithmetic Sequence,

$$A_n = A_0 + nd$$

$$x_n = A_n 2^n = (A_0 + nd)2^n.$$

Substituting $n = 0$ and 1, we get

$$1 = x_0 = A_0$$

$$4 = x_1 = (A_0 + d) \times 2$$

So $A_0 = d = 1$, and so the solution to (3) is

$$x_n = (1+n)2^n$$

There are many other applications of difference equations to other problems. For example, suppose you drew n lines across a sheet of paper in such a way that any 2 lines intersect and no 3 lines pass through the same point. If R_n is the number of resulting regions then

(a) if you draw no lines there is one region (the whole sheet) and so $R_0 = 1$;

(b) if you have already divided the sheet into R_n regions with n lines and draw another line, then each intersection creates a new region (before reaching it), and one new region is created after the last intersection (try it and see!), i.e.

$$R_{n+1} = R_n + n + 1.$$

We leave you the problem of solving this difference equation!

PYTHAGORAS' THEOREM REVISITED

Peter Donovan

Consider the following three theorems from Euclid's Elements.

I.49 In right-angled triangles the square on the side subtending the right angle is equal to the squares on the sides containing the right angle.

VI.8 If in a right-angled triangle a perpendicular be drawn from the right angle to the base the triangles adjoining the perpendicular are similar both to the whole and to one another.

VI.19 Similar triangles are to one another in the duplicate ratio of the corresponding sides.

Euclid had no reason to deduce I.49 from VI.8 and VI.19. It was not until 1908 that M.A. Naber of Moorn, Holland noted that it is easy to do so. As Euclid does not use I.49 to prove VI.8 or VI.19, such a deduction is valid.

Can you discover this simple proof of Pythagoras' Theorem given that in modern parlance VI.19 is stating that the ratio of the areas of similar triangles is the ratio of the squares of the lengths of corresponding sides?