Parabola Volume 29, Issue 1 (1993)

ECONOMICS, THE COBWEB MODEL AND APPLIED MATHEMATICIANS

Alex Opie*

When overseas economists tackle the problem of supply and demand they like to talk about pig–iron, hogs and corn. As we live in Australia let us discuss the production and buying of wheat. If growers try to sell (supply) more than is wanted (demand) the price per kilogram falls. The growers then decide to produce less for next year's sale which may lead to a shortage and the price will rise. The buyers don't like the higher price and buy less, that is, demand falls and so on ....

To make planning easier both sellers and buyers might want smaller jumps and drops in the price — a more stable price. Applied mathematicians like to look at such real–world problems and make a mathematical model of the system: the model must be simple enough so that they can solve the mathematics yet contain enough of the underlying problem so that the solution is useful. You will notice that I started the paragraph with economists and ended with applied mathematicians – well, economists apply mathematics.

We will need to define symbols and use them to write equations.

The number of kilograms of wheat needed in 1993, or the demand in 1993, will be called D_{1993}; so the demand in year k is D_k. Now demand goes up as the price, p_k, in year k goes down. A guess at the way this works is shown on the graph below.

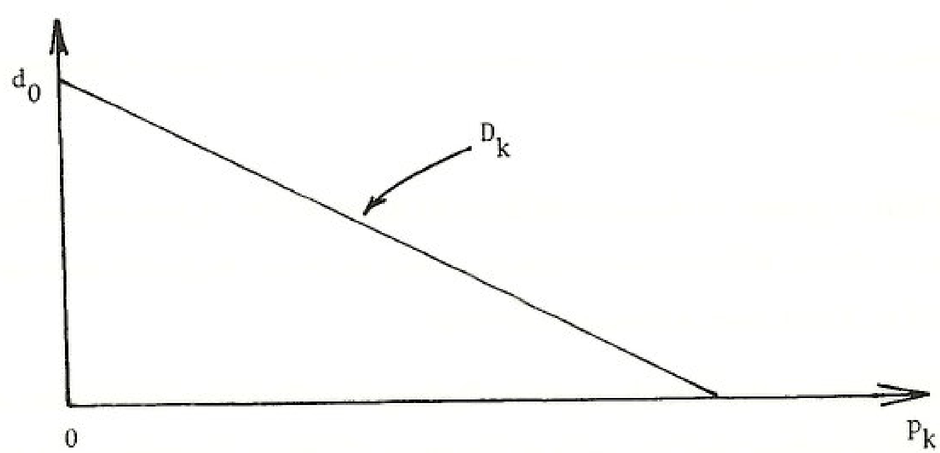

Fig.1 A straight–line demand/price curve.

* Alex is an Applied Mathematician at the University of New South Wales

Our guess shows D_k values forming a straight line, the slope of which is very significant, as we shall see.

The demand when the price is zero is d_0: even when it is free, people only need so much wheat. The equation which describes this graph is

$$D_k = d_0 - gp_k \tag{1}$$

Here the minus sign ensures that as price (p_k) goes up demand (D_k) goes down. The symbol g indicates how steep the slope of the line will be and it will be a positive number.

With similar reasoning we can write down an equation for the number of kilograms of wheat grown or supply, S_k, in year k. This will be large if the price last year p_{k-1} was large, and small if p_{k-1} was small. A straight line guess for the graph is now shown (Fig.2) and the equation would look like

$$S_k = b(p_{k-1} - P) \tag{2}$$

$$(\text{or } S_k = bp_{k-1} - bP = bp_{k-1} - c).$$

Here it can be seen that if the price at year $k-1$ drops to P the growers decide that it is not worthwhile to supply any wheat in year k and S_k drops to zero.

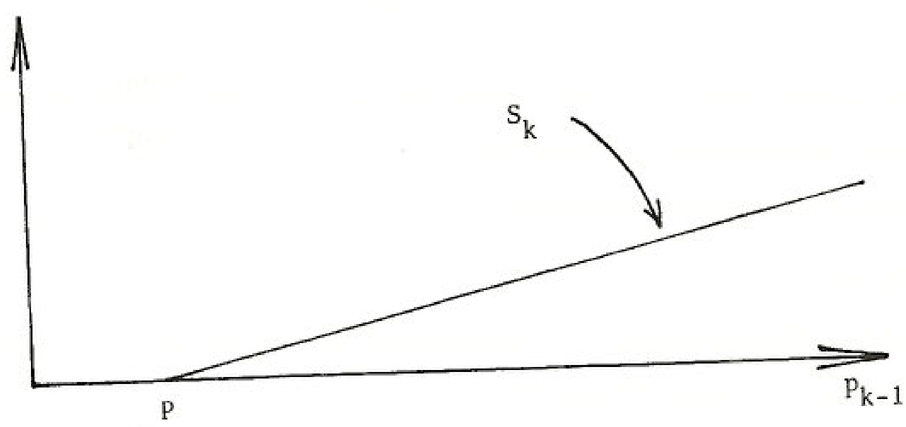

Fig.2 A straight–line supply/price curve.

The size of the positive number b determines the steepness of the S_k line.

Exercise 1:

Suppose

$$D_k = 1 - 2p_k, \qquad \text{and}$$

$$S_k = (p_{k-1} - \frac{1}{2})$$

(i) Draw the graph of D_k carefully on graph paper.

(ii) Draw the graph of S_k carefully, perhaps on a sheet of tissue paper, using the same scales as used in (i).

Of course what really happens at a market is that the price settles to a value so that all the wheat put on sale is bought. This means that p_k can be worked out by just setting the expression for the supply at year k equal to the expression for the demand at year k, i.e. set the right side of (1) equal to the right side of (2)

$$d_0 - gp_k = b(p_{k-1} - P) \tag{3}$$

So, if we know what the graphs look like (the values of d, g, b and P) and last year's price (p_{k-1}) we can work out this year's price (p_k).

An interesting feature of (3) is that it works for any two years $k-1$ and k.

Exercise 2:

Suppose that the price at year $k-1$ is the same as at year k. Use (3) to derive the following formula for this price (call it p^*):

$$p^* = \frac{d_0 + bP}{g + b} \tag{4}$$

This special price p^*, which stays the same year after year, is referred to as the equilibrium price and if it can be achieved will ensure that supply and demand stay equal to each other and constant year after year.

Exercise 3:

Find formulae for supply (S^*) and demand (D^*) if they stay constant, in equilibrium, year after year. Compare S^* and D^*.

If we place the graphs of Figures 1 and 2 over the top of each other, as in Figure 3, we can see some of the features we have just talked about.

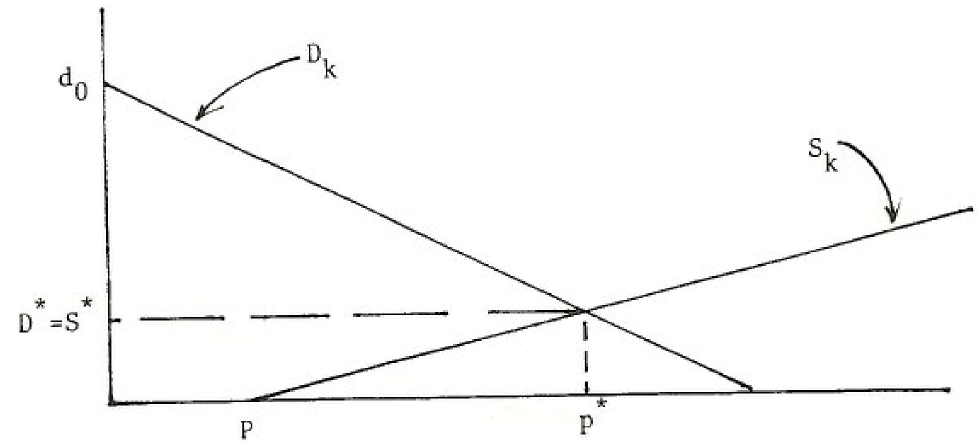

Fig.3 Straight–line supply and demand curves.

Notice that D_k is the same as S_k where they cross. This is just the equilibrium value of D_k and S_k, call it D^* or S^*. The crossing point occurs when p_k takes the value p^*. So, if we have the graphs we don't have to do the algebra of Exercises 1 and 2, we just read off the graph.

Exercise 4:

Suppose, as in Exercise 1, that

$$D_k = 1 - 2p_k$$
$$S_k = (p_{k-1} - \frac{1}{2})$$

(i) Use algebra to find p^* and D^*.

(ii) Using the graphs of Exercise 1 lay the S_k graph on top of the D_k graph and read off the values of p^* and D^*. Compare your answers with those of (i).

The glut and famine "cycles" described at the beginning need not just repeat. We could imagine that things could settle down to a situation where supply and demand gradually approach each other over the years. On the other hand the cycles could become so wild that one year the growers produce no wheat: they either pack up for good and walk off their land or, for the same effect, all the buyers of wheat die or leave for lack of food. The question is: given the graphs of Figures 1,2 and 3 can we decide what will happen?

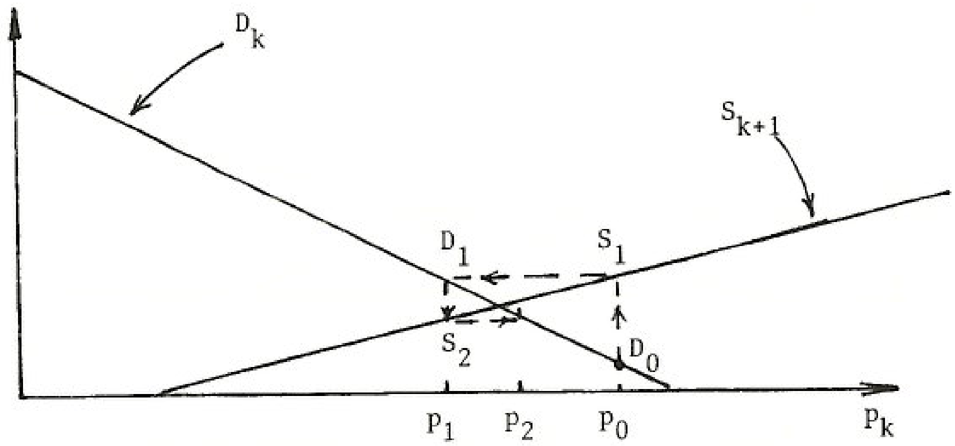

Fig.4. Supply/Demand Cobweb

Consider Figure 4. At the market one year (say the year 0) a price p_0 is obtained for each kilogram of wheat and the demand is D_0. This price p_0 will determine the supply S_1 at next year's market. Remember that, by formula (2), we have $S_1 = b(p_0 - P)$.

So the price p_0 sets the value S_1 which is found by going straight up from the D_0 dot to the supply curve at S_1 (arrowed line).

The next step uses the fact that, at the year 1 market, the price p_1 is set so that demand equals supply i.e. the supply S_1 is the same as the demand D_1 and this is shown by moving horizontally from S_1 to D_1. Now, as before, the price at year 1 determines the supply at year 2 so we use p_1 to find S_2 on the supply curve: graphically we go vertically down from D_1 to the point marked S_2 i.e. this is the S_2 one calculates using p_1. The last step in the cycle is found by seeing that the supply S_2 must be the same as the demand D_2 for clearance of wheat: that is $S_2 = D_2$ which determines the price p_2 and we go horizontally from S_2 on the supply graph to D_2 on the demand graph to find p_2

One can then keep threading horizontally and vertically, a special cobweb, all the time bringing the S points and D points closer to each other while the p–values jump back and forth in smaller and smaller steps. We are approaching equilibrium.

Exercise 5:

The formulae in Exercise 1 and 4 may be written as

$$D_k = 1 - 2p_k$$
$$S_{k+1} = (p_k - \frac{1}{2}).$$

Plot them one the same piece of graph paper and use the cobweb construction to investigate the approach to equilibrium.

Exercise 6:

Repeat Exercise 5 but use

$$D_k = 1 - 2p_k$$
$$S_{k+1} = 4(p_k - \frac{1}{2})$$

(The slope of the S_{k+1}-line has been increased.)

Exercise 7:

Repeat Exercise 5 but use

$$D_k = 1 - 2p_k$$
$$S_{k+1} = 2(p_k - \frac{1}{2})$$

Exercise 8:

(i) Show that Equation (3) can be written as

$$p_k = -\frac{b}{g}p_{k-1} + \frac{d_0}{g} + \frac{b}{g}P$$

(ii) Using the value of the constants given in Exercise 4 and starting with $p_0 = 4$ use your calculator to find $p_1, p_2, p_3, \ldots$.

(iii) Repeat (ii) for Exercise 6.

(iv) Repeat (ii) for Exercise 7.

(v) Are these results telling you the same things as Exercise 5, 6 and 7.

Exercise 9:

For the situations seen in Exercises 6 and 7 what would you think about the chance of achieving constant prices over the years?

Exercise 10: (Hard. You will probably need more tests.)

Can you guess the rule which determines whether equilibrium is approached?

Conclusion:

In order to achieve a simple model we have had to assume that the supply and demand curves were straight lines. (Many other assumptions have been made — can you name any?) Even if this was the case we also have to know the constants d, g, b and P; but these could be found by watching a market over the years and observing D_k, S_k and p_k for $k = 0, 1, 2, \ldots$. In fact such observations would tell us if the straight–line assumption was a good one, at least in the short–term (i.e. over a few years).

The point is, our model was a guess: it needs experimental evidence to support it. Only then could we hope to use it to make predictions and these predictions had better not be for too far ahead. What is more we would have to take care using our model on a different market — we would have to do the experiments all over again to make sure different influences had not appeared.

An applied mathematician builds mathematical models which can be used to predict but, to verify the predictions, observation is essential. If observation does not agree with prediction the model must be made better or may even have to be junked and a new one tried.

ADDENDUM:

Clearly the farmers would do better if they took account of more prices rather than just last year's: this would lead to an Equation 3 with additional terms, $c\,p_{k-2} + d\,p_{k-3} + \ldots$. Such equations are known as recurrence relations of order 1 (Equation 3), order 2, order 3, ... and they can be solved using systematic methods. One might also like to include terms which are powers of $p_{k-1}, p_{k-2}, \ldots$ and such equations are called non–linear (or higher degree). We might also like to include other products in our market: for example, we need iron to make farm implements so another recurrence relation which relates prices for wheat and iron would be necessary. In general we could have a set of nth order, non–linear recurrence relation. Of these varied modifications only a non–linear term is needed for the appearance of chaos. By this I mean that a small difference in starting conditions, say a minutely different starting demand, would lead to totally different cobwebs and perhaps a production wipe–out rather than an approach to equilibrium.