SOLUTIONS OF PROBLEMS 872-881

Q.872 A flea on the number line jumps from the point a to the point b, given by $a + \frac{1}{b} = 1$. If the flea keeps jumping according to the same rule, show that it comes back to its original position eventually. In how many jumps? (Assume $a \neq 0,\ a \neq 1$.)

ANSWER After one jump the flea is at $b = \frac{1}{1-a}$.

Note that since $a \neq 0,\ b \neq 1$, and since $a \neq 1,\ b$ is defined; also $b \neq 0$. After two jumps, the flea is at

$$c = \frac{1}{1-b} = \frac{1}{1-\frac{1}{1-a}} = \frac{1-a}{-a} = \frac{a-1}{a}.$$

Since $b \neq 0, 1,\ c$ is defined and not 0,1. After three jumps the flea is at

$$d = \frac{1}{1-c} = \frac{1}{1-\frac{a-1}{a}} = a.$$

In other words, after three jumps the flea is back where it started!

Q.873 In my laundry, the ceiling is laid out in squares, like the number plane. There is a light at the centre, which we will think of as the origin. A moth flits about on the ceiling according to the following rule:

If it is at (x, y), it flits to $(y, y-x)$. Show that it eventually comes back to where it was. How many flights does it make? What shape of path does it take? Consider the brightness of the light at (x, y), given by $b = 1/(x^2 - xy + y^2)$. What extra information does this give you about the path the moth follows?

ANSWER After one flight the moth is at $(y, y-x)$;

after two, at $(y-x, (y-x)-y) = (y-x, -x)$;

after three, at $(-x, -x-(y-x) = (-x, -y)$;

after four, at $(-y, x-y)$;

after five, at $(x-y, x)$;

after six, at (x, y), back where it started.

The path is a hexagon. At each of the six points on the path the brightness b is the same, since

$$x^2 - xy + y^2 = y^2 - (y-x)y + (y-x)^2.$$

This shows that all six points lie on the curve $x^2 - xy + y^2 =$ constant, which is an ellipse centred on the origin.

Q.874 A person can climb a staircase one or two steps at a time. In how many different ways can (s)he climb a staircase of 10 steps? of 100 steps? of n steps? (There are 5 ways of climbing a staircase of 4 steps: $1,1,1,1; 1,1,2; 1,2,1; 2,1,1; 2,2$.) What if (s)he climbs 2 or 3 at a time?

ANSWER Let u_n be the number of ways of climbing a staircase of n steps 1 or 2 steps at a time. Then $u_4 = 5$, as shown in Parabola Vol.28 Number 2 p.43.

Indeed $u_1 = 1,\ u_2 = 2,\ u_3 = 3,\ u_4 = 5$, and so on.

The sequence $\{u_n\} = \{1, 2, 3, 5, \cdots\}$.

It is a fact that $u_{n+2} = u_{n+1} + u_n$ for $n \geq 1$, or, $u_n = u_{n-1} + u_{n-2}$ for $n \geq 3$. That is, each term in the sequence $\{u_n\}$ is the sum of the preceding two terms. To see this, consider climbing to the top of an n-step staircase. Your last "step" was either one or two steps. If it was one step, you could have got to the second top step in u_{n-1} ways, before taking your last "step", while if it was two steps, you could have got to the third top step in u_{n-2} ways before taking your last "step".

In any case, if you look at the solution to Problem 875, you will see that if we can find two numbers r, s such that $r + s = 1,\ rs = -1$, we will be able to say

$$u_{n+2} = (r+s)u_{n+1} - rsu_n,$$

and a formula for u_n will be

$$u_n = Ar^n + Bs^n.$$

If $r + s = 1,\ rs = -1$, we can find r, s by solving the quadratic equation

$$x^2 - (r+s)x + rs = 0,$$

$$\text{or,}\quad x^2 - x - 1 = 0.$$

The solution is

$$x = \frac{1 \pm \sqrt{5}}{2}.$$

We can take $r = \dfrac{1+\sqrt{5}}{2}$, $s = \dfrac{1-\sqrt{5}}{2}$.

Then $u_n = A\left(\dfrac{1+\sqrt{5}}{2}\right)^n + B\left(\dfrac{1-\sqrt{5}}{2}\right)^n$.

To find A, B, we use the fact that $u_1 = 1$, $u_2 = 2$. We solve

$$A\left(\frac{1+\sqrt{5}}{2}\right) + B\left(\frac{1-\sqrt{5}}{2}\right) = 1,$$

$$A\left(\frac{1+\sqrt{5}}{2}\right)^2 + B\left(\frac{1-\sqrt{5}}{2}\right)^2 = 2.$$

These equations give

$$A = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right), \quad B = -\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right).$$

It follows that

$$u_n = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right\}.$$

This is our formula for u_n. To find u_{10} or u_{100}, we need only substitute and use our calculator or a computer. I found that

$$u_{10} = 89$$

$$u_{100} = 573147844013817084101$$

Going back to our formula for u_n, the quantity $\dfrac{1}{\sqrt{5}}\left(\dfrac{1-\sqrt{5}}{2}\right)^n$ is small, and gets smaller as n gets larger. We can ignore it and say that

$$u_n \text{ is the integer closest to } \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}.$$

This is quite a simple formula!

The second problem, where you are allowed to take two or three steps at a time is much harder. In this case, $u_{n+3} = u_{n+1} + u_n$ for $n \geq 1$, with $u_1 = 0$,

$u_2 = 1,\ u_3 = 1$. The formula I found is

for $n > 1, u_n$ is the closest integer to $\frac{a+1}{2a+3}a^n$, where

$$a = \sqrt[3]{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{23}{27}}} + \sqrt[3]{\frac{1}{2} - \frac{1}{2}\sqrt{\frac{23}{27}}},$$

or, u_n is the closest integer to

$$(0.41149558866264576338) \times (1.32471795724474602596)^n.$$

This gives $u_{10} = 7,\ u_{100} = 670976837021$.

Q.875 A sequence $\{u_n\}$ is given by the formula

$$u_n = Ar^n + Bs^n$$

for fixed numbers A, B, r, s. Show that for every n,

$$u_{n+2} = (r+s)u_{n+1} - rsu_n$$

Given that $u_1 = 1,\ u_2 = 2,\ u_3 = 3,\ u_4 = 5$, find A, B, r, s and hence find a formula for u_n.

ANSWER $u_n = Ar^n + Bs^n$

$u_{n+2} - (r+s)u_{n+1} + rsu_n$

$$= (Ar^{n+2} + Bs^{n+2}) - (r+s)(Ar^{n+1} + Bs^{n+1}) + rs(Ar^n + Bs^n)$$

$$+ Ar^n[r^2 - (r+s)r + rs] + Bs^n[s^2 - (r+s)s + rs]$$

$$= Ar^n[0] + Bs^n[0]$$

$$= 0.$$

$$u_1 = 1,\ u_2 = 2,\ u_3 = (r+s)u_2 - rsu_1 = 2(r+s) - rs = 3,$$

$$u_4 = (r+s)u_3 - rsu_2 = 3(r+s) - 2rs = 5.$$

We have to solve

$$2(r+s) - rs = 3$$

$$3(r+s) - 2rs = 5.$$

Doubling the first and subtracting the second gives $r + s = 1$.

3 times the first minus twice the second gives $rs = -1$.

This is precisely the situation in Problem 874, and the solution is

$$u_n = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right\}.$$

This problem was solved by Katy Lai, James Ruse A.H.S.

Q.876 Given a set of $2n + 1$ numbers in arithmetic progression, in how many ways can 3 distinct numbers also in arithmetic progression be selected?

ANSWER There are an odd number of numbers, $a_0, \cdots, a_{2n}$ spaced d apart. We are going to choose three of them, $a < b < c$.

The middle one, b, can be any of $a_1, \cdots, a_{2n-1}$.

If $b = a_1$, there is only one choice for a, namely $a = a_0$ (and then $c = a_2$).

If $b = a_2$, there are two choices for a, namely $a = a_0 (c = a_4)$ or $a = a_1 (c = a_3)$.

(I am assuming for the moment that $n \geq 2$, but the formula I arrive at also works for $n = 1$.)

If $b = a_k$, $k < n$ (in other words if b is less than halfway along the a's), there are k choices for a (and for c). If $b = a_n$, there are n choices for a (and for c).

If $b = a_k$ with $k > n$, the number of choices for c is $2n - k$. This is true right up to when $b = a_{2n-1}$, and there is one choice for c (and for a). The total number of ways of choosing three different numbers in arithmetic progression is therefore

$$1 + 2 + \cdots + (n-1) + n + (n-1) + \cdots + 1 = n^2.$$

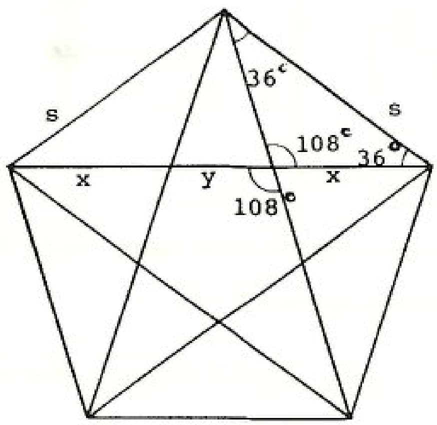
$$[1 + 2 + \cdots + n = \frac{1}{2}n(n+1),$$

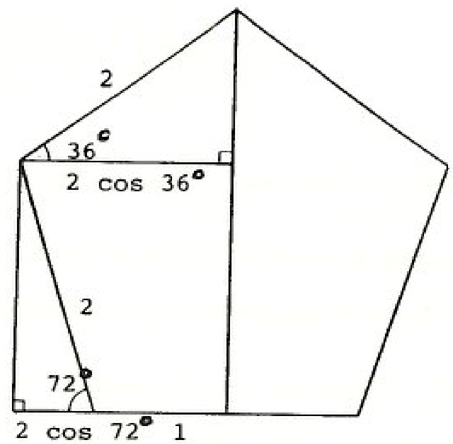
$$1 + 2 + \cdots + (n-1) = \frac{1}{2}n(n-1),$$

$$\frac{1}{2}n(n+1) + \frac{1}{2}n(n-1) = n^2.]$$

Q.877 A regular pentagon has all its diagonals drawn in; there is a small pentagonal region in the centre. What is the ratio of the area of the smaller pentagon to the larger? Try to give your answer in terms of $\sqrt{5}$.

ANSWER





From the diagram,

$$\frac{2x+y}{\sin 108^\circ} = \frac{s}{\sin 36^\circ}$$
$$\frac{x}{\sin 36^\circ} = \frac{s}{\sin 108^\circ}$$
$$2x+y = s\frac{\sin 72^\circ}{\sin 36^\circ}$$
$$x = s\frac{\sin 36^\circ}{\sin 72^\circ}$$
$$y = s\left(\frac{\sin 72^\circ}{\sin 36^\circ} - 2\frac{\sin 36^\circ}{\sin 72^\circ}\right).$$

$$\begin{aligned}\frac{y}{s} &= \frac{\sin 72^\circ}{\sin 36^\circ} - 2\frac{\sin 36^\circ}{\sin 72^\circ} \\ &= 2\cos 36^\circ - \frac{1}{\cos 36^\circ} \\ &= \frac{2\cos^2 36^\circ - 1}{\cos 36^\circ} \\ &= \frac{\cos 72^\circ}{\cos 36^\circ}\end{aligned}$$

From the second diagram

$$2\cos 36^\circ = 1 + 2\cos 72^\circ = 1 + 2(2\cos^2 36^\circ - 1).$$

$$4\cos^2 36^\circ - 2\cos 36^\circ - 1 = 0.$$

$$\cos 36^\circ = \frac{1+\sqrt{5}}{4}$$

$$\cos 72^\circ = 2\left(\frac{1+\sqrt{5}}{4}\right)^2 - 1 = \frac{\sqrt{5}-1}{4}$$

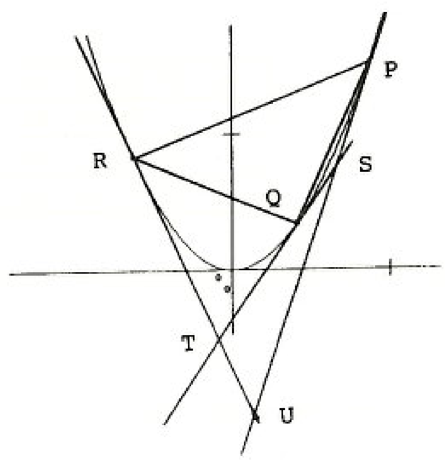
$$\frac{y}{s} = \frac{\cos 72^\circ}{\cos 36^\circ} = \frac{\sqrt{5}-1}{\sqrt{5}+1} = \frac{3-\sqrt{5}}{2}$$

$$\frac{\text{Area of small pentagon}}{\text{Area of large pentagon}} = \frac{y^2}{s^2} = \frac{7-3\sqrt{5}}{2}.$$

Solved by Katy Lai, James Ruse A.H.S.

Q.878 Consider the parabola $x^2 = 4ay$, and let P, Q, R be three points on it. The three tangents to the parabola at P, Q, R meet in pairs at points S, T, U. Prove that the area of $\triangle STU$ is half that of $\triangle PQR$.

ANSWER



Suppose $P = (2ap, ap^2)$, $Q = (2aq, aq^2)$, $R = (2aq, ar^2)$.

The tangent at P is $y - ap^2 = p(x - 2ap)$,

or, $px - y = ap^2$

The tangent at Q is $qx - y = aq^2$

so $S = (a(p+q),\ apq)$.

Similarly, $T = (a(q+r),\ aqr)$, $U = (a(r+p),\ apr)$.

The area of the triangle with vertices (x_1, y_1), (x_2, y_2), (x_3, y_3) is

$$\frac{1}{2}|x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3|.$$

(or in the language of determinants,

$$\text{area} = \frac{1}{2}\left|\det\begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix}\right|.)$$

The area of $\triangle PQR$ is thus easily calculated to be

$$a^2|(p-q)(q-r)(r-p)|$$

and that of $\triangle STU$ to be

$$\frac{1}{2}a^2|(p-q)(q-r)(r-p)|.$$

Q.879 A four-by-four block of one-by-one squares has a number in each of the sixteen squares, in such a way that the numbers in each row, each column and each diagonal have the same sum, k say. Show that the numbers in the four corners sum to k.

a_1	a_2	a_3	a_4
b_1	b_2	b_3	b_4
c_1	c_2	c_3	c_4
d_1	d_2	d_3	d_4

ANSWER

Let the numbers in the first row be a_1, a_2, a_3, a_4, in the second b_1, b_2, b_3, b_4 and so on.

Then

$$a_1 + a_2 + a_3 + a_4 = s$$

$$b_1 + b_2 + b_3 + b_4 = s$$

$$c_1 + c_2 + c_3 + c_4 = s$$

$$d_1 + d_2 + d_3 + d_4 = s$$

$$a_1 + b_1 + c_1 + d_1 = s$$

$$a_2 + b_2 + c_2 + d_2 = s$$

$$a_3 + b_3 + c_3 + d_3 = s$$

$$a_4 + b_4 + c_4 + d_4 = s$$

$$a_1 + b_2 + c_3 + d_4 = s$$

and $$a_4 + b_3 + c_2 + d_1 = s$$

Then $a_1 + a_4 + d_1 + d_4 =$

$= (a_1 + b_2 + c_3 + d_4) + (a_4 + b_3 + c_2 + d_1) - (b_2 + b_3 + c_2 + c_3)$

$= 2s - (b_2 + b_3 + c_2 + c_3)$

Also $a_1 + a_4 + d_1 + d_4 =$

$= (a_1 + \cdots + a_4) + (d_1 + \cdots + d_4) - (a_2 + b_2 + c_2 + d_2) - (a_3 + b_3 + c_3 + d_3)$

$+(b_2 + b_3 + c_2 + c_3)$

$= b_2 + b_3 + c_2 + c_3$

It follows that $a_1 + a_4 + d_1 + d_4 = s$

(and that $b_2 + b_3 + c_2 + c_3 = s$).

Solved by Katy Lai, James Ruse A.H.S.

Q.880 The number x satisfies the equation

$$x^2 = \sqrt{2}x + \sqrt[3]{3}.$$

Show that x satisfies a polynomial equation with **rational** coefficients. (We call such a number "**algebraic**".)

ANSWER $x^2 = \sqrt{2}x + \sqrt[3]{3}$

So $(x^2 - \sqrt{2}x)^3 = 3$

or $x^6 - 3\sqrt{2}x^5 + 6x^4 - 2\sqrt{2}x^3 = 3$,

$x^6 + 6x^4 - 3 = 3\sqrt{2}x^5 + 2\sqrt{2}x^3$,

$(x^6 + 6x^4 - 3)^2 = 2(3x^5 + 2x^3)^2$,

$x^{12} + 12x^{10} + 36x^8 - 6x^6 - 36x^4 + 9 = 2(9x^{10} + 12x^8 + 4x^6)$,

so finally,

$x^{12} - 6x^{10} + 12x^8 - 14x^6 - 36x^4 + 9 = 0.$

Q.881 If you have seventeen points in space and each pair is joined by an interval coloured either red, green or blue, show that there is a red, a green or a blue triangle.

ANSWER Consider one of the seventeen points; P_0 is joined to 16 others by red, green or blue intervals. At least six of these are of the one colour. Consider a set of six points joined to P_0 by intervals of the same colour, say red. If two of these six points are joined to each other by a red interval, we have a red triangle.

If not, then every pair of the six are joined by green or blue intervals. Choose one of the six points, P_1. P_1 is joined to each of the other five by green or blue intervals. At least three of these intervals are the one colour. Consider a set of three points joined to P_1 by intervals of the same colour, say green. If two of the three points are joined by a green interval, we have a green triangle.

If not, then every pair of the three are joined by blue intervals, and we have a blue triangle.