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UNSW SCHOOL MATHEMATICS COMPETITION 1993

SOLUTIONS

JUNIOR DIVISION

1. Given real numbers a, b with $0 < a \leq b$, a sequence of numbers is defined by

$$x_1 = a\,, \quad x_2 = b\,, \quad x_{n+2} = |x_{n+1}| - x_n \text{ for } n \geq 1.$$

Find x_{1993}.

Solution. We have

$$x_1 = a\,,\ x_2 = b\,,\ x_3 = b - a\,,\ x_4 = (b-a) - b = -a\,,\ x_5 = a - (b-a) = 2a - b\,.$$

Now if $2a \geq b$ then

$$\begin{aligned}
x_6 &= (2a-b) - (-a) = 3a - b \\
x_7 &= (3a-b) - (2a-b) = a && \text{since } 3a - b > 2a - b \geq 0 \\
x_8 &= a - (3a-b) = b - 2a \\
x_9 &= (2a-b) - a = a - b && \text{since } b - 2a \leq 0 \\
x_{10} &= (b-a) - (b-2a) = a && \text{since } a - b \leq 0 \\
x_{11} &= a - (a-b) = b\,;
\end{aligned}$$

while if $2a < b$ then

$$\begin{aligned}
x_6 &= (b-2a) - (-a) = b - a \\
x_7 &= (b-a) - (2a-b) = 2b - 3a \\
x_8 &= (2b-3a) - (b-a) = b - 2a \\
& && \text{since } 2b - 3a > 2b - 4a > 0 \\
x_9 &= (b-2a) - (2b-3a) = a - b && \text{since } b - 2a > 0 \\
x_{10} &= (b-a) - (b-2a) = a \\
x_{11} &= a - (a-b) = b\,.
\end{aligned}$$

Note that in either case, $x_{10} = x_1$ and $x_{11} = x_2$. Since each term in the sequence depends only on the previous two terms, it follows that $x_{12} = x_3$, $x_{13} = x_4$, and in general $x_n = x_{n-9}$ whenever $n \geq 10$. Consequently, noting that $1993 = 9 \times 221 + 4$, we find

$$x_{1993} = x_{1984} = x_{1975} = \cdots = x_4 = -a \ .$$

2. Find a solution in positive integers of the equation

$$x^2 + xy - y = 1993 \ .$$

What is the total number of such solutions?

Solution. Subtracting 1 from each side of the equation and factorising the left hand side gives

$$(x-1)(x+y+1) = 1992 \ .$$

Now $1992 = 2^3 \times 3 \times 83$, so 1992 can be written as a product of two positive factors in the following ways:

$$\begin{aligned} 1992 &= 1 \times 1992 = 2 \times 996 = 3 \times 664 = 4 \times 498 \\ &= 6 \times 332 = 8 \times 249 = 12 \times 166 = 24 \times 83 \ . \end{aligned}$$

Clearly $x - 1$ must be the smaller factor and $x + y + 1$ the larger, so each of the above expressions $1992 = ab$ yields one and only one solution

$$x = a + 1 \ , \quad y = b - x - 1 = b - a - 2 \ .$$

Thus there eight solutions.

x	2	3	4	5	7	9	13	25
y	1989	992	659	492	324	239	152	57

3. Reconstruct the following long multiplication, in which each dot represents a square

digit (0, 1, 4 or 9), and the question mark represents any digit.

```
      . . . .
    × . . . .
      -------
      . . . .
    . . . . .
  . . . . .
. . 6 . .
-------------
2 . 5 . . ? . .
```

Solution. Since there are only three non-zero square digits, the fourth and fifth lines must be equal, and different from the sixth. Also, the first digit of the sixth line must be 1, and hence the entire sixth line is at most 19699. Now suppose that the first digit of the second line is a 9. Then the fifth line would be at most $\frac{4}{9}$ of the sixth line, that is, at most 8755, which is impossible since the fifth line has five digits. Therefore the second line is 4991.

Now the first digit of the first line must be 4, for if it were 1 or 9 then multiplying by 4 would not give a five-digit number beginning with 1 for the sixth line. Therefore the sixth line is at least $4000 \times 4 = 16000$. Indeed, since the second digit is a square and the third is 6, the sixth line is at least 19600; and so the first line is at least a quarter of this, that is, 4900. Moreover, since the sixth line is at most 19699, the first is at most 4924. If the last digit of the first line were 4 or 9 then the last digit of the sixth line would be 6, which is not a square. Therefore we have reduced the possibilities for the first line to four: 4900, 4901, 4910 or 4911. Trying each of these separately shows that only 4911 gives a product consistent with the given information in the seventh line. Thus the multiplication is as follows.

```
        4 9 1 1
    ×   4 9 9 1
        -------
        4 9 1 1
      4 4 1 9 9
    4 4 1 9 9
  1 9 6 4 4
  -------------
  2 4 5 1 0 8 0 1
```

4. Let n be a positive integer. Show that the number of sequences of ones and threes adding up to $n+1$ is equal to the number of sequences of ones and twos, with no adjacent twos, adding up to n. Note that order is important in a sequence. For example, if $n = 5$ then the possible sequences of the second type are

$$212,\ 2111,\ 1211,\ 1121,\ 1112,\ 11111.$$

Solution. For each sequence of the first type we may produce a sequence of the second type by the following rule: replace each three by a one and a two, in that order; then delete the first number of the sequence (which must be a one). Clearly the resulting sequence will have no adjacent twos, for every two will either be the first element of the sequence or will be preceded by a one. Also, the sum of the numbers in the original sequence was $n+1$, and we have deleted a one, so the sum in the derived sequence is n. This confirms that we have indeed created a sequence of the second type. The procedure can be reversed: given the sequence we have just found, put an extra one at the beginning; then every two in the sequence is preceded by a one, and we may substitute a three for each pair one-two (in that order). This gives a sequence of the first type, indeed, the very one we started with.

We have shown that the sequences of the first and second types can be matched up in pairs; therefore the numbers of sequences of each type are equal.

5. Divide the numbers $24, 38, 39, 44, 45, 46, 48$ into two sets in such a way that the sum of the numbers in each set is prime. Show that this can be done in only one way.

Solution. The list contains just two odd numbers. These numbers must be placed in different sets, as otherwise the two sums would be even and therefore not prime. So we begin by setting

$$S = \{39\}, \quad T = \{45\}.$$

If all seven numbers are divided by 3 the remainders are $0, 2, 0, 2, 0, 1, 0$ respectively. In order to avoid having one or the other sum divisible by 3, the numbers with remainder 2 must both be in one set, the number with remainder 1 in the other. There are two possibilities so far:

$$S = \{38, 39, 44\}, \ T = \{45, 46\} \quad \text{or} \quad S = \{39, 46\}, \ T = \{38, 44, 45\}.$$

In the first case the sum of S is 121 and the sum of T is 91, neither of which is prime. So we must add 24 to one set and 48 to the other; however, adding 24 to *either* gives a total which is divisible by 5 and therefore not prime. Hence the first case does not lead to a solution.

In the second case S and T add up to 85 and 127 respectively. Since 85 is not prime, S must also contain the number 24 or 48 or both. If S contains 48 only its sum becomes 133, which is divisible by 7; if S contains 24 only then T has sum $127 + 48 = 175$, which is divisible by 5. Therefore S must contain both numbers, and we have the only possible solution

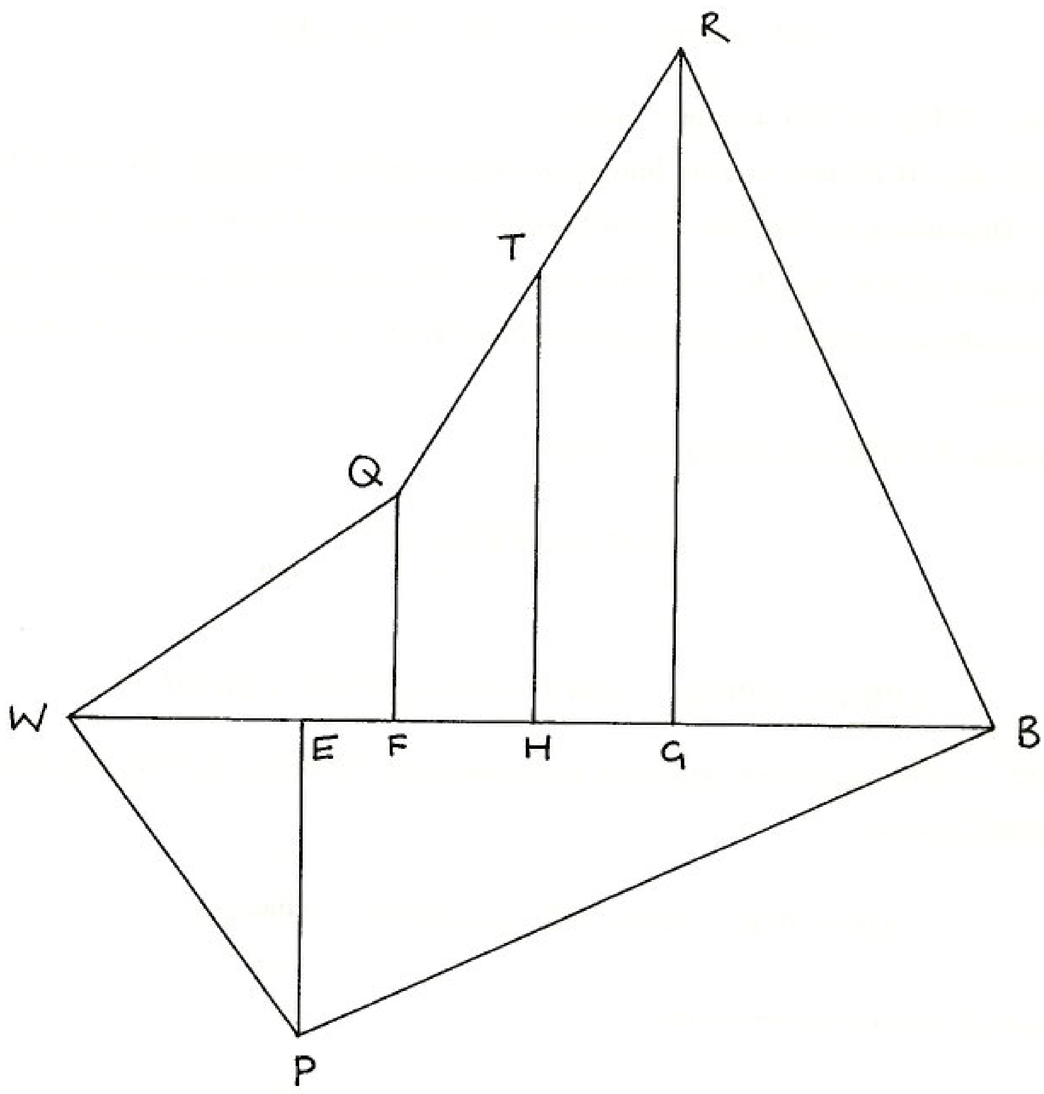
$$S = \{24, 39, 46, 48\}, \quad T = \{38, 44, 45\}$$

with sums 157 and 127, which you may check are both prime.

6. An old manuscript reads as follows: "...Having reached the island, walk from the palm tree to the white rock, turn 90° right, and walk the same distance as you have just walked (that is, from the tree to the rock). Place a peg in the ground. Return to the palm tree, walk to the black rock, turn 90° left, and walk a distance equal to that from the tree to the black rock. Place another peg in the ground. Dig for the treasure half way between the pegs." When you arrive, you find that the rocks are

easily identifiable, but many more trees have grown up and it is impossible to tell which one was meant. Can you find the treasure?

Solution.



In the above diagram W and B represent the white and black rocks, P the palm tree, Q and R the first and second pegs and T the location of the treasure. Thus it is given that

$$PW = WQ \ , \quad PB = BR \ , \quad TR = TQ$$

and that $\angle PWQ$, $\angle PBR$ are right angles.

Draw the line WB, and to this line draw perpendiculars PE, QF, RG and TH as shown. Depending on the exact location of P with respect to W and B, the order of the points E, F, G and H may differ from the diagram (indeed, some of the points may coincide, and it may be necessary to extend WB), but a similar proof will apply in all cases.

In triangles $\triangle PWE$ and $\triangle WQF$ we have

$$\angle PEW = \angle WFQ = 90°$$

and

$$\angle PWE = \angle PWQ - \angle EWQ = 90° - \angle FWQ = \angle WQF \ ;$$

also $PW = WQ$, so the two triangles are congruent. Similarly $\triangle PBE$ is congruent to $\triangle BRG$. Hence

$$QF = WE \ , \quad RG = EB \quad \text{and} \quad WF = PE = BG \ .$$

To locate T we first observe that

$$TH = \tfrac{1}{2}(QF + RG) = \tfrac{1}{2}(WE + EB) = \tfrac{1}{2}WB \ .$$

Also, since $QT = RT$ and the lines QF, TH, RG are parallel, we have $FH = GH$. Therefore

$$WH = WF + FH = BG + GH = BH \ .$$

Thus to locate the treasure you need only walk from the white rock halfway towards the black rock, turn 90° left and walk a further distance equal to this, then get out your spade and start digging!

SENIOR DIVISION

1. Divide the numbers 24, 38, 39, 44, 45, 46, 48 into two sets in such a way that the sum of the numbers in each set is prime. Show that this can be done in only one way.

   Solution. See question 5 in the Junior Division.

2. An old manuscript reads as follows: "...Having reached the island, walk from the palm tree to the white rock, turn 90° right, and walk the same distance as you have just walked (that is, from the tree to the rock). Place a peg in the ground. Return to the palm tree, walk to the black rock, turn 90° left, and walk a distance equal to that from the tree to the black rock. Place another peg in the ground. Dig for the treasure half way between the pegs." When you arrive, you find that the rocks are easily identifiable, but many more trees have grown up and it is impossible to tell which one was meant. Can you find the treasure?

   Solution. See question 6 in the Junior Division.

3. If a, b, c are integers and

$$a + b + c = -1 \ , \quad a^2 + b^2 + c^2 = 1993 \ ,$$

find the numerator when

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$$

is reduced to lowest terms.

Solution. Firstly,

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{bc + ca + ab}{abc} = \frac{\frac{1}{2}\left((a+b+c)^2 - (a^2+b^2+c^2)\right)}{abc} = -\frac{996}{abc} \ .$$

The minus sign could be included either in the numerator or in the denominator; and we need to find out what, if anything, can be cancelled from the numerator of ± 996. Since $a + b + c$ is odd, either two or none of the numbers a, b, c are even. Suppose none is even. Then we can write

$$\begin{aligned} a^2 + b^2 + c^2 &= (2A+1)^2 + (2B+1)^2 + (2C+1)^2 \\ &= 4(A^2 + B^2 + C^2 + A + B + C) + 3 \ , \end{aligned}$$

and so $a^2 + b^2 + c^2$ leaves remainder 3 when divided by 4. But this is impossible since in fact 1993 leaves remainder 1 when divided by 4. Therefore two of the numbers a, b, c must be even, and we can cancel 4 from the numerator and denominator of the fraction, leaving ± 249 for the numerator.

Similarly, one of the three integers must be divisible by 3. For otherwise we would have

$$a^2 + b^2 + c^2 = (3A \pm 1)^2 + (3B \pm 1)^2 + (3C \pm 1)^2$$
$$= 3(3A^2 + 3B^2 + 3C^2 \pm 2A \pm 2B \pm 2C + 1) ,$$

once again impossible since 1993 is not a multiple of 3. Therefore, cancelling 3, we can reduce the numerator to ± 83. Finally,

$$a^2 \leq 1993 < 83^2 ,$$

so 83 is not a factor of a (nor, similarly, of b or of c). Hence no further cancellation is possible, and the fraction has been reduced to lowest terms. The numerator is ± 83.

If you would like to check this, you can find by trial and error that the possible values of the fraction are

$$\tfrac{1}{36} - \tfrac{1}{21} - \tfrac{1}{16} = -\tfrac{83}{1008} , \quad -\tfrac{1}{36} + \tfrac{1}{24} + \tfrac{1}{11} = \tfrac{83}{792} \quad \text{and} \quad -\tfrac{1}{33} + \tfrac{1}{30} + \tfrac{1}{2} = \tfrac{83}{165} .$$

4. A salesman travels round and round a circuit of n towns, where $n \geq 2$, spending exactly one day at a time in each. From town 1 he travels to 2, from 2 to 3, and so on, except that from town n he has the option of going to 1 as usual, or skipping 1 and going immediately to 2. If you know this, and also know that he starts in town 1 on day 1, what is the first day on which the salesman's location is *completely* unknown to you, that is, the first day on which, as far as you know, he might be in any one of the n towns? Prove your answer.

Solution. Imagine not one salesman but two travelling around the circuit. One travels as slowly as possible, always going from town n to town 1; the other, as quickly as possible, always going from town n to town 2. "Our" salesman could be anywhere between these two limits, and his location will be *completely* unknown when the fast salesman has caught up to the town just behind the slow one. This will happen *for* *the first time* on day d (say), when the fast salesman has just skipped a town: that is, when he is in town 2 and the slow traveller is in town 3. Now the slow traveller will be in town 3 on day 3, and every n days thereafter, so

$$d = 3 + xn$$

where x is the number of circuits made by the slow salesman. The fast salesman completes a circuit in $n-1$ days, and so $d = 2 + y(n-1)$. Equating these expressions and rearranging,

$$yn - xn = y + 1.$$

Therefore n is a factor of $y+1$, and for the smallest possible solution we take $y = n-1$. This yields $d = 2 + (n-1)^2$. So the first day on which the salesman could be in any one of the n towns is day $n^2 - 2n + 3$.

5. So much has Tom Black aged in appearance, it is hard to believe he is still under fifty. I knew, though, that his recent illness would not have quenched his enthusiasm for a puzzle, so I said to him, "Tom, my favourite football team has played three times this season. The total number of points scored in the three games is exactly equal to your age, and the product of the three numbers is 1260. Can you work out the number of points scored in each game?" He did a little calculating, but pronounced that he was unable to decide between two possibilities. "Well," said I, "one of the numbers is greater than my age. If you know my age, that information will be enough to settle the problem." "As a matter of fact," Tom replied, "I didn't know your age. But I do now."

   How old am I? (Ages are in whole years.)

   Solution. Since Tom presumably knows his own age, we seek to express 1260 as a product of three factors in two ways, such that the sum of the factors in each expression is the same (and less than 50). By trial and error we find the following four possibilities:

$$36 + 7 + 5 = 35 + 9 + 4 = 48\ ,$$
$$35 + 6 + 6 = 30 + 14 + 3 = 47\ ,$$
$$21 + 10 + 6 = 18 + 14 + 5 = 37\ ,$$
$$18 + 10 + 7 = 15 + 14 + 6 = 35\ .$$

Note that although we do not know which of the four possible sums is the correct one, Tom does. (Hints for finding the factorisations: note that $1260 = 2 \times 2 \times 3 \times 3 \times 5 \times 7$ and that the largest of the three factors is less than 50. We easily see that $49, 48, 47, 46$ are not factors of 1260. Now 45 is a factor, but the total of all three factors will be at least $45 + 7 + 4$ which is too big. So we consider 44 and 43, give a brief glance at 42, and so on until we reach 36, which yields the first triple given above.)

Now, how could the knowledge of "my" age help Tom to solve the problem? The only answer is that he is thinking of two triples such that my age lies between the greatest number in one triple and the greatest number in the other. For example, if Tom were 47 years old and I were 32, he could deduce that the scores were not $30, 14, 3$ but $35, 6, 6$. However, Tom said that he did not at first know my age, but deduced it: he could not have done so in the example just given, as the problem could have been solved equally well had my age been $34, 33, 32, 31$ or 30.

The only situation in which Tom could have deduced my age is where there is *only one* number that is less than the maximum number in one triple, but not less than the maximum in the other; that is, where the two maximum numbers differ by 1. Thus the first case is the correct one, and I am 35 years old. (And Tom is 48, and the scores were $36, 7, 5$.)

6. Twenty one 3×1 rectangles are placed without overlapping on a normal 8×8 chessboard, thus covering 63 of the 64 squares. Determine all possible locations of the uncovered square.

   Solution. Starting at the left-hand end, label the top row of the chessboard with the numbers 1,2,3,1,2,3,1,2. Similarly, label the second row 2,3,1,2,3,1,2,3, the third 3,1,2,3,1,2,3,1, the fourth row in the same way as the first, the fifth in the same way as the second and so on. It is easy to see that wherever a 3×1 rectangle is placed on the board it must cover a 1, a 2 and a 3. But counting up all the figures we find that 1 and 3 occur twenty-one times each while 2 occurs twenty-two times. Therefore the uncovered square must contain the figure 2.

   Now add some more labels to the chessboard: A,B,C,A,B,C,A,B in the first *column* (starting at the top), C,A,B,C,A,B,C,A in the second column, B,C,A,B,C,A,B,C in

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