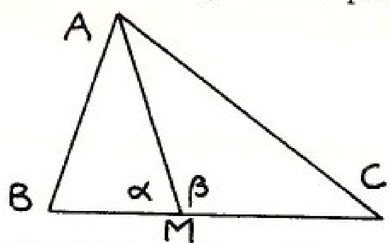
## SOLUTIONS TO PROBLEMS 882-892

**Q.882** A triangle is divided by one straight line into two parts which are similar to each other. Prove that the triangle is isosceles or right-angled (or both).

**ANS.** Clearly the line must pass through a vertex of the triangle, for otherwise the two parts are a triangle and a quadrilateral, which certainly are not similar.




Now (see diagram) since $\triangle ABM$ and $\triangle ACM$ are similar, one of the angles of $\triangle ACM$ must equal $\alpha$. If this angle is not $\beta$, then $\triangle ACM$ has the two angles $\alpha$ and $\beta$ adding up to 180°; so the third angle must be zero, which is impossible. Thus $\alpha = \beta = 90°$. Now $\angle ABM$ is equal to either $\angle ACM$ or $\angle MAC$ (or both, if $\angle ABM = 45°$); and $\triangle ABC$ is isosceles in the first case, right-angled in the second.

**Q.883** A number consists of the nine digits $1, 2, \ldots, 9$ once each (in some order). The last digit is 5. Prove that the number is not a square.

**ANS.** Suppose the number is a square, say $N^2$. Since $N^2$ ends in 5, $N$ must also end in 5. Let $d$ be the tens digit of $N$, and $M$ the number consisting of all the previous digits of $N$: thus $N = 100M + 10d + 5$. Expanding the square and collecting terms we have

$$N^2 = 1000(10M^2 + 2Md + M) + 100(d^2 + d) + 25.$$

Hence the last two digits of $N^2$ are 25, and the hundreds digit equals the last digit of $d^2 + d$. Trying $d = 0, 1, \cdots, 9$ we find this digit to be $0, 2, 6, 2, 0, 0, 2, 6, 2, 0$ respectively. But under the conditions of the problem 0 and 2 are impossible, so $N^2$ ends in 625 and $d = 2$ or 7. Now let $c$ be the hundreds digit of $N$, and $L$ the previous digits, so that

$$N = 1000L + 100c + 25 \quad \text{or} \quad 1000L + 100c + 75.$$

In the first case we have

$$N^2 = 10000(100L^2 + 20Lc + c^2 + 5L) + 1000(5c) + 625.$$

Thus the thousands digit of $N^2$ is 0 if $c$ is even, 5 if $c$ is odd; but both of these must be disallowed. Similarly, if $N = 1000L + 100c + 75$ then

$$N^2 = 10000(100L^2 + 20Lc + c^2 + 15L) + 1000(15c + 5) + 625$$

and the thousands digit is 5 if $c$ is even, 0 if $c$ is odd, which again is impossible. So we have ruled out all possibilities, and we conclude that the number cannot be a square.

**Correct solution:** Lisa Gotley, All Saints Anglican School, Merrimac, Queensland.

**Q.884** On a clock face, six adjacent numbers are left untouched and the other six are rearranged so that all around the clock face, every pair of consecutive numbers adds up to a prime number. What is the final arrangement of numbers?

**ANS.** Adding pairs of numbers all round the clock face gives the twelve sums

$$3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 13.$$

If six adjacent numbers are to be left untouched we must find five primes in a row already in this sequence. The only possibility is $23, 13, 3, 5, 7$; and so the unchanged numbers are $11, 12, 1, 2, 3, 4$. We now rearrange the remaining six numbers: keeping in mind that obviously even and odd numbers must alternate, we quickly arrive at the two solutions:

$$1, 2, 3, 4, 7, 10, 9, 8, 5, 6, 11, 12$$

$$1, 2, 3, 4, 9, 10, 7, 6, 5, 8, 11, 12.$$

**Correct solution:** Tammy Beshay, St. Patrick's College.

**Q.885** Prove (without extensive calculations!) that

$$\frac{31}{45} < \frac{1000}{1001} \times \frac{1002}{1003} \times \frac{1004}{1005} \times \cdots \times \frac{1992}{1993} < \frac{23}{31}.$$

**ANS.** Note that for any $n \geq 0$,

$$\frac{n}{n+1} < \frac{n+1}{n+2}.$$

Writing $P$ for the given product, we have

$$\begin{aligned}
P^2 &= \frac{1000}{1001} \times \frac{1000}{1001} \times \frac{1002}{1003} \times \frac{1002}{1003} \times \cdots \times \frac{1992}{1993} \times \frac{1992}{1993} \\
&< \frac{1000}{1001} \times \frac{1001}{1002} \times \frac{1002}{1003} \times \frac{1003}{1004} \times \cdots \times \frac{1992}{1993} \times \frac{1993}{1994} \\
&= \frac{1000}{1994}
\end{aligned}$$

since almost everything cancels. The job can now be finished by calculator, or more elegantly

$$P^2 < \frac{1000}{1994} = \frac{500}{997} < \frac{529}{961} = \left(\frac{23}{31}\right)^2.$$

Similarly

$$\begin{aligned}
P^2 &> \frac{999}{1000} \times \frac{1000}{1001} \times \frac{1001}{1002} \times \frac{1002}{1003} \times \cdots \times \frac{1991}{1992} \times \frac{1992}{1993} \\
&= \frac{999}{1993} > \frac{961}{2025} = \left(\frac{31}{45}\right)^2.
\end{aligned}$$

Hence

$$\frac{31}{45} < P < \frac{23}{31}.$$

**Q.886** Find three prime numbers $a, b, c$, all different, such that

$$a^2 + 37ab = c^3 + 1656.$$

**ANS.** Can $c$ equal 2? If so, then

$$a(a + 37b) = 1664 = 2^7 \times 13.$$

Since $a$ is prime and $a \neq c$, we have $a = 13$, $a + 37b = 128$; but this cannot be, since $b$ is an integer. Hence $c$ is odd. Since the product of $a$ and $a + 37b$ is odd, each factor must be odd; so their diference $37b$ is even; so $b$ is even. The only even prime is 2. Substituting $b = 2$ into the original equation,

$$a^2 + 74a = c^3 + 1656;$$

rearranging and then factorising a quadratic, we get

$$(a-18)(a+92)=c^3.$$

Now since $c$ is prime, $c^3$ has only two factorisations as a product of two numbers,

$$c^3 = 1 \times c^3 = c \times c^2.$$

Clearly $a-18$ is the smaller factor, so we have

$$a-18=1,\ a+92=c^3$$

or

$$a-18=c,\ a+92=c^2.$$

The first of these gives $c^3 = 111$ which is impossible, while the second yields the quadratic $c^2 = c + 110$ and leads to the unique solution

$$a=29,\ b=2,\ c=11.$$

**Very good solution:** Lisa Gotley, All Saints Anglican School, Merrimac, Queensland.

**Q.887** Last year was a busy one for my family, with six of my brothers and sisters having children. Writing myself a timetable for buying niecely and nephewly birthday presents, I noticed some odd facts about the six dates. The difference (in days) between two consecutive birthdays was always the same, and this difference was a prime number. No two children were born in consecutive months, and no two were born on the same date in different months. One of the birthdays was on August 8. When were the others?

**ANS.** Let $n$ be the number of days difference between each birthday and the previous one. Since there are six birthdays, with one in August and no two in consecutive months, there must be one in October and one in December. Thus a gap of $2n$ days must be at least from August 8 to December 1 (115 days), and at most from

August 8 to December 31 (145 days). Hence (remembering that $n$ is a whole number)

$$58 \leq n \leq 72.$$

As $n$ is prime we have the four possibilities $n = 59, 61, 67$ or $71$. Now we need to calculate intervals of $n$ days each side of August 8 and see which $n$ give admissible answers. A convenient way to do this is to count the days of all months from January to July,

$$31 + 28 + 31 + 30 + 31 + 30 + 31 = 212,$$

to see that August 8 is the 220th day of the year. So if $n = 59$ we consider days

$$43, 102, 161, 220, 279, 338;$$

by a similar method to that above, these days are

Feb 12, Apr 12, Jun 10, Aug 8, Oct 6, Dec 4.

But this is not allowed since a birthday would fall on the same date in both February and April. Trying $n = 61, 67, 71$ gives respectively

| | | | | | |
|---|---|---|---|---|---|
| Feb 6, | Apr 8, | Jun 8, | Aug 8, | Oct 8, | Dec 8; |
| Jan 19, | Mar 27, | Jun 2, | Aug 8, | Oct 14, | Dec 20; |
| Jan 7, | Mar 19, | May 29, | Aug 8, | Oct 18, | Dec 28; |

of which the first must be discarded. So, two solutions? ... well, if you read the question carefully, I said that all this happened *last year*, so we should have counted not 28 but 29 days for February! Repeating the working, we find the values of $n$ unchanged, and the birthdays corresponding to $n = 59, 61, 67, 71$ are

| | | | | | |
|---|---|---|---|---|---|
| Feb 13, | Apr 12, | Jun 10, | Aug 8, | Oct 6, | Dec 4; |
| Feb 7, | Apr 8, | Jun 8, | Aug 8, | Oct 8, | Dec 8; |
| Jan 20, | Mar 27, | Jun 2, | Aug 8, | Oct 14, | Dec 20; |
| Jan 8, | Mar 19, | May29, | Aug 8, | Oct 18, | Dec 28. |

Only the first is a solution to the problem. (By the way, if the given birthday had been August 7, the above dates would have been shifted back one day; but this would also have opened up the possibility that $n = 73$ and given a second solution: Jan 1, Mar 14, May 26, Aug 7, Oct 19, Dec 31 – just made it!)

**Q.888** Alexander, David, Esther, Jacinda and Simon all received different marks in the maths test which was held unexpectedly last week. In the following, students who made correct statements invariably had obtained higher marks than those who made incorrect statements.

| | |
|---|---|
| Simon: | Alexander and Esther gained the top two places. |
| Jacinda: | No, what Simon just said is wrong. |
| David: | I was ranked in between Simon and Jacinda. |
| Alexander: | Jacinda came second. |
| Jacinda: | I scored fewer marks than Esther. |
| Esther: | Exactly three of the previous five statements are correct. |

Find the order in which the students finished.

**ANS.** Suppose that Simon's statement is correct. Then Alexander came higher up the list than Simon and therefore must also have spoken correctly. But this is impossible since it would mean that Alexander, Esther and Jacinda each occupy one of the top two places. Thus Simon must have been wrong. This means that Jacinda's first statement is true, and so her second statement must be true too. Thus Esther came ahead of Jacinda. To summarise what we know so far, (part of) the order of marks is

$$\ldots \text{ Esther } \ldots \text{ Jacinda } \ldots \text{ Simon } \ldots$$

and Jacinda made two correct statements, Simon one incorrect statement. Since three of the first five statements are true, we see that of Alexander's and David's remarks one is true and one false. If David was correct and Alexander incorrect, then David came below Jacinda, and so did Alexander (since his statement was false); thus Jacinda came second and Alexander's statement was true after all. Thus David must have made a false statement and finished last, while Alexander made a true statement and came third. So the order (top down) was

Esther, Jacinda, Alexander, Simon, David.

**Partial solution:** Lisa Gotley, All Saints Anglican School, Merrimac, Qld.

**Q.889** Find all positive integers $m$ such that if

$$(1993+m)^{1993}$$

is expanded by the Binomial Theorem, two adjacent terms are equal.

**ANS.** The $k$th and $(k+1)$th terms in the expansion are equal when

$$\binom{1993}{k} 1993^k m^{1993-k} = \binom{1993}{k+1} 1993^{k+1} m^{1993-k-1},$$

that is,

$$\frac{1993!}{k!(1993-k)!} 1993^k m^{1993-k} = \frac{1993!}{(k+1)!(1992-k)!} 1993^{k+1} m^{1992-k}.$$

After much cancellation this becomes

$$\frac{m}{1993-k} = \frac{1993}{k+1}$$

which can be rearranged to give

$$(m+1993)k = 1993^2 - m;$$

hence

$$(m+1993)(k+1) = 1993^2 + 1993 = 1994 \times 1993$$
$$= 2 \times 997 \times 1993,$$

where 2, 997 and 1993 are all prime. Thus $m+1993$ is a factor of $2\times 997\times 1993$, and clearly $m+1993 \geq 1994$. There are eight factors, of which $1, 2$, 997 and 1993 are rejected as too small, leaving four solutions

$$m + 1993 = 2\times 997,\quad 2\times 1993,\quad 997\times 1993 \quad \text{or} \quad 2\times 997\times 1993,$$

that is,

$$m = 1,\ 1993,\ 1985028 \quad \text{or} \quad 3972049.$$

**Q.890** If $n$ is a positive integer, we define $N(n)$ to be the number of ways of writing

$$n = x_0 + 2x_1 + 2^2x_2 + 2^3x_3 + \cdots$$

where $x_0, x_1, \ldots$ may take the values $0, 1, 2$ or $3$. For example, $N(9) = 5$ since

$$\begin{aligned} 9 &= 1 + (2 \times 0) + (2^2 \times 0) + (2^3 \times 1) \\ &= 1 + (2 \times 0) + (2^2 \times 2) \\ &= 1 + (2 \times 2) + (2^2 \times 1) \\ &= 3 + (2 \times 1) + (2^2 \times 1) \\ &= 3 + (2 \times 3) \end{aligned}$$

are the five ways of writing 9 in the given form. Find a formula giving $N(n)$ in terms of $n$.

ANS. Experimenting a bit first, we find

$$\begin{aligned} &1 = 1 \\ &2 = 0 + (2 \times 1) = 2 \\ &3 = 1 + (2 \times 1) = 3 \\ &4 = 0 + (2 \times 0) + (2^2 \times 1) = 0 + (2 \times 2) = 2 + (2 \times 1) \\ &5 = 1 + (2 \times 0) + (2^2 \times 1) = 1 + (2 \times 2) = 3 + (2 \times 1), \end{aligned}$$

so that

$$N(1) = 1,\ N(2) = 2,\ N(3) = 2,\ N(4) = 3,\ N(5) = 3.$$

A reasonable guess would be

$$N(n) = \begin{cases} \frac{1}{2}\,(n+2) & \text{if } n \text{ is even} \\ \frac{1}{2}\,(n+1) & \text{if } n \text{ is odd.} \end{cases} \qquad (*)$$

We shall prove this by an extended version of mathematical induction. For the basis step, we have already seen that the formula is true for $n = 1, 2$ and $3$ (and more!) Before going on to the inductive step we note the following. Let $n$ be an even number, $n = 2k$, with $k \geq 2$. If

$$n = 2k = x_0 + 2x_1 + 2^2 x_2 + \cdots$$

then $x_0 = 0$ or 2. If $x_0 = 0$ then

$$k = x_1 + 2x_2 + 2^2x_3 + \cdots$$

while if $x_0 = 2$ then

$$k - 1 = x_1 + 2x_2 + 2^2x_3 + \cdots$$

So the number of ways of writing $2k$ is the number of ways of writing $k$, plus the number of ways of writing $k-1$. That is,

$$N(2k) = N(k) + N(k-1) \quad \text{if } k \geq 2.$$

Similarly, if $n = 2k+1$ is odd, then $x_0 = 1$ or 3 and we find

$$N(2k+1) = N(k) + N(k-1) \quad \text{if } k \geq 2.$$

Now we proceed with the inductive step. Assume that $(*)$ is true for all values of $n$ less than some even number $2k$, where $k \geq 2$. Then

$$\begin{aligned} N(2k) &= N(k) + N(k-1) \\ &= \begin{cases} \frac{1}{2}\,(k+2) + \frac{1}{2}k & \text{if } k \text{ is even} \\ \frac{1}{2}\,(k+1) + \frac{1}{2}(k+1) & \text{if } k \text{ is odd} \end{cases} \\ &= \tfrac{1}{2}(2k+2), \end{aligned}$$

so $(*)$ is true for $n = 2k$. Also

$$N(2k+1) = N(2k) = \tfrac{1}{2}(2k+2) = \tfrac{1}{2}((2k+1)+1)$$

which shows that $(*)$ is true for $n = 2k+1$. This completes the proof.

(A bit more explanation on this kind of induction: we knew early on that the formula is true for $n < 4$. Then the inductive step shows that it is true for the next two values of $n$ as well, that is, for $n < 6$. Using the inductive step again verifies the result for $n < 8$, and so forth.)

**Q.891** Andrew is given a bag of lollies by his parents and told to share them with his little sister Becky. The number of lollies in the bag is not known, but is somewhere from 1 to 100. Being just a little bit greedy, Andrew shares out the lollies according to the scheme "one for you, two for me, three for you, four for me, ... " Any leftover lollies at the end go to the person who would have received them anyway. Thus if there are eleven lollies, then Becky gets one, Andrew gets two, Becky gets three, Andrew gets four, and Becky gets the last one. After all the lollies are shared out, how far ahead, on average, can Andrew expect to be?

ANS. If there is one lolly in the bag, Becky gets it and Andrew's advantage is $-1$. If there are two, the children receive one each and Andrew's advantage is 0. If the bag contains a third lolly, Andrew gets it and his advantage is now $+1$. Continuing in this way, if there are $1, 2, \ldots, 100$ lollies in the bag, Andrew's advantage will be respectively

$$-1, 0, 1, 0, -1, -2, -1, 0, 1, 2, \ldots, 2.$$

(Note the pattern here: the list starts with $-1$, then increases twice, decreases three times, increases four times and so on.) To find out how far ahead Andrew can expect to be on average we must add up all these numbers and divide by 100. A relatively easy way to do this is to split the list into groups of $1, 2, 3, \ldots, 13$ numbers, with an "incomplete" fourteenth group consisting of the remaining nine numbers, and then add each group:

$$\begin{aligned} -1 &= -1 \\ 0 + 1 &= 1 \\ 0 - 1 - 2 &= -3 \\ -1 + 0 + 1 + 2 &= 2 \\ 1 + 0 - 1 - 2 - 3 &= -5 \\ -2 - 1 + 0 + 1 + 2 + 3 &= 3 \\ \vdots \quad & \quad \vdots \\ -6 - 5 - 4 - 3 - 2 - 1 + 0 + 1 + 2 &= -18. \end{aligned}$$

The sum of the first thirteen groups is

$$-1 + 1 - 3 + 2 - 5 + 3 - 7 + 4 - 9 + 5 - 11 + 6 - 13 = -28;$$

adding the incomplete group and dividing by 100, Andrew's expected advantage is

$$(-28 - 18)/100 = -0.46.$$

In other words Andrew should expect to end up *behind* by about half a lolly! The explanation for this is that Andrew's extra lolly per "round" is more than outweighed by Becky's expectation of receiving the leftover lollies at the end.

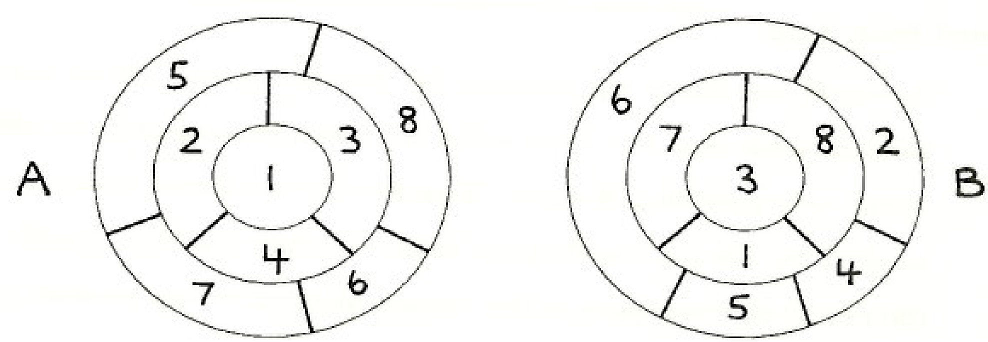
An alternative method would be, using a similar process to the above, to calculate the total number of lollies each child will receive on average. If you do this you will find the result Andrew 25.02, Becky 25.48, which confirms the difference calculated above.

**Q.892** A certain race of beings lives on two planets. Each nation owns one connected piece of territory on each planet. It is desired to colour a map (pair of maps?) of the two planets according to two rules. First, two countries with a common border must bear different colours; second, each nation must have its two territories (one on each planet) coloured with the same colour.

(i) Find an example of a pair of maps which requires eight colours.

(ii) (Probably difficult.) Can you find an example where nine colours are needed?

**ANS.** (i) A bit of experimentation, inspired by the four-country configuration given last issue, leads to one possible pair of maps as shown.




The numbers represent the different nations. It is easy to check that any one of the eight countries has a border with each of the other seven. For example, country 7 shares borders with 2,4,5 and 6 on planet $A$, and with 1,3,6 (again) and

8 on planet $B$. Therefore no two countries may use the same colour, and eight colours are needed.

**(ii)** I haven't found any pair of maps where nine colours are necessary. However my source of information says that such maps can be produced; and that on the other hand, twelve colours are enough to solve the problem for any given maps. If you can find out anything about this please write to us!

**Continued from p.24**

**Q.901** In the mythical country of Ozz there are three kinds of coins: in order of increasing value the cong, the dong and the fong. Two Ozzians, Alice and Bert, each with his or her own pile of congs, dongs and fongs, play the following game. Alice chooses any one of her coins, whereupon Bert must take from his pile one each of the other two kinds. They then toss the three coins and add up the value of those which fall heads; the player whose total is greater takes all three coins. There are no ties except when all three coins turn up tails, in which case each player keeps his or her coin(s). Alice and Bert notice that in the long run neither player has any advantage, regardless of which coin Alice selects. How many congs are in a fong?

**Q.902** Given three non-collinear points $X, Y, Z$ such that $\angle XYZ$ is obtuse, show how to construct a triangle $\triangle ABC$ such that the median of the triangle through $A$ intersects the circumcircle at $X$, the angle bisector through $A$ intersects the circumcircle at $Y$, and the altitude through $A$ intersects the circumcircle at $Z$.

**Continued from p.22**

the third, the fourth the same as the first and so on. (Note that the second and third columns may not be in quite the order you expect!) As above, the letter A occurs once more than B and C. Therefore the uncovered square must be one of those labelled 2A. See the first diagram below. To show that it is actually possible to place the twenty-one rectangles in the manner required we can produce the solution in the second diagram.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1A | 2C | 3B | 1A | 2C | 3B | 1A | 2C |
| 2B | 3A | 1C | 2B | 3A | 1C | 2B | 3A |
| 3C | 1B | 2A | 3C | 1B | 2A | 3C | 1B |
| 1A | 2C | 3B | 1A | 2C | 3B | 1A | 2C |
| 2B | 3A | 1C | 2B | 3A | 1C | 2B | 3A |
| 3C | 1B | 2A | 3C | 1B | 2A | 3C | 1B |
| 1A | 2C | 3B | 1A | 2C | 3B | 1A | 2C |
| 2B | 3A | 1C | 2B | 3A | 1C | 2B | 3A |