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MATHEMATICS AND PIGEON HOLES

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Before reading the rest of this article try to prove the following statements.

(1) If a drawer contains a large number of socks of the same colour but of two different sizes, and I take out three socks, then two of these will make a pair.

(2) If a school has 750 students there is at least one day of the year when three (or more) students all have their birthday.

(3) There are two people in Sydney who (i) were born in the same year; (ii) have the same height (within one tenth of a millimetre); and (iii) have the same weight (within a kilogram).

(4) In any party of n persons we can always find two people who have shaken hands with the same number of other people in the party.

(5) If $t_1, t_2, \cdots, t_{10}$ are any ten integers, then there are some values of i and j with $1 \leq i \leq j \leq 10$ such that 10 divides $t_i + t_{i+1} + \cdots + t_j$ exactly.

If you have succeeded in proving the five statements above, then you will probably have discovered that there is a basic similarity in their proofs. This similarity can be formalized to what is known as the Pigeon Hole Principle (PHP) of Dirichlet:

If we have m objects distributed among n boxes ("pigeon holes") then at least one box has at least m/n objects in it. In particular, if we have n objects in n boxes, and no box has two objects in it, then each box has exactly one object in it.

This simple and almost trivial observation has many far reaching applications in mathematics. It turns out to be the core of the proof of many useful and important theorems. The rest of this article is devoted to some simple examples.

We shall first see how the PHP proves (5). Put $s_0 = 0$, $s_1 = t_1$, $s_2 = t_1 + t_2, \cdots, s_{10} = t_1 + t_2 + \cdots t_{10}$. After division by 10, each s_i will have remainder r_i with $0 \leq r_i \leq 9$. Now

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put the 11 different t_i into 10 "boxes" labelled $0, 1, 2, \cdots 9$ where s_i goes into box j if $r_i = j$. By the PHP, there are two s_i in one of the boxes. Thus we may suppose that s_m and s_n, say, with $m < n$, both lie in box k. That is, the remainders r_m and r_n both equal k. Then 10 divides $s_n - s_m = t_{m+1} + t_{m+2} + \cdots + t_n$.

Similarly, we have

(6) If p is a prime number, and n is any integer not divisible by p, then there is an integer m with $1 \leq m \leq p-1$ such that p divides $mn - 1$.

To prove this consider the $p-1$ numbers $n, 2n, 3n, \cdots, (p-1)n$. Since p does not divide i or n, p does not divide ni. Therefore the remainder r_i left after ni is divided by p is one of the $p-1$ numbers $j = 1, 2, \cdots, p-1$. We also notice that if $r_i = r_k$, then p divides $ni - nk = n(i-k)$. Since p does not divide n, p must divide $(i-k)$. But i and k lie between 1 and $p-1$, so their difference is less than p. Therefore $i - k$ equals $0.p$, i.e. $i = k$. This shows that all the remainders r_i are different. If we put ni into a "box" labelled j when $r_i = j$, then we have $p-1$ objects (the ni) in $p-1$ boxes, and at most one object lies in each box. Therefore the PHP shows that each box contains exactly one object. In particular, $r_m = 1$ for some m, and then p divides $mn-1$.

You will have noticed that the PHP can tell us that something exists (such as the integer m in (6)), but it gives us no help in finding it. For this reason, proofs using the PHP are often referred to as **existence proofs** in contrast to proofs which are **constructive.**

I conclude with an important "approximation" theorem which tells us that we can approximate any real number quite closely by a rational number. Try to prove it using the PHP.

Let x be any real number and N be any integer ≥ 1. Then there exists a rational number p/q (with p and q integers and $0 < q \leq N$) such that the difference $|x - p/q|$ is less than $1/Nq$. (For example, if $x = \pi$ and $N = 10$, we can take $p/q = 22/7$. Then $|\pi - 22/7| < 1/70$).