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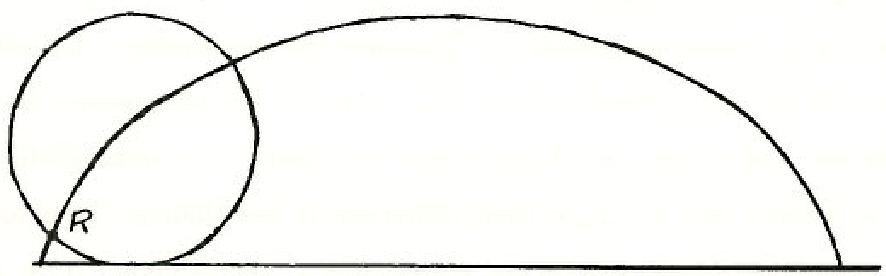
THE CYCLOID: "THE HELEN OF GEOMETERS"

David Tacon

Consider a fixed reflector R on the rim of a bicycle wheel of radius r. If the wheel rolls along a straight line without slipping the reflector R traces out a curve called a **cycloid**. Probably the cycloid has caught your attention when you have observed some cyclist riding along at night. It was named by Galileo Galilei (1564-1642) after the Greek word meaning circle.

Figure 1



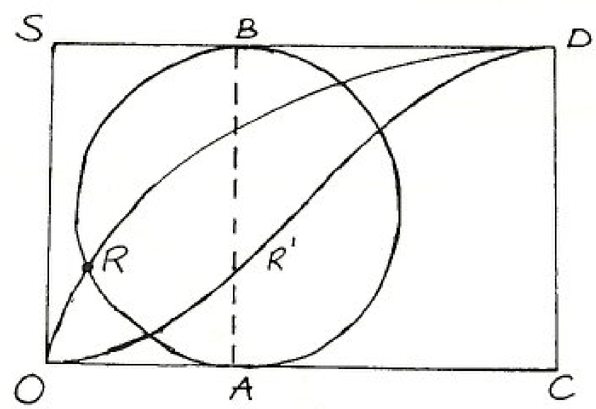
Indeed the history of the curve really only dates back to the time of Galileo. What is of interest to us however is the contents of a letter that Galileo wrote to his friend Bonaventura Cavalieri (1598-1647) in 1640. Galileo wrote:

"More than fifty years ago the curved line came to my mind and I wanted to describe it, admiring it because of its gracious curvature, adaptable to the arches of a bridge. I made several tentative calculations on it and on the space comprised between it and its chord, in order to demonstrate some property. **And it seemed at first that such space may be three times the circle which it describes, but it was not that."**

In fact Galileo's first hunch about the area was correct and the proof of this fact is what we primarily wish to discuss here. As part of our story it is interesting to note that in about 1630 the French monk Marin Mersenne (1588-1648) suggested using the cycloid as a test curve for the different methods of dealing with areas via "infinitesimals". We need keep in mind here that we are speaking pre-Newton and the development of the Calculus so a straightforward integration was out of the question. In fact in Mersenne's time no scientific journals existed and Mersenne fulfilled a central role in the history of mathematics as a skilled communicator and disseminator of knowledge. Mersenne played this role admirably since he could understand new discoveries quickly and could pose questions clearly and well. (He also had great moral character so that his several hundred correspondents trusted him.) In any case the cycloid became one of the most discussed curves of the period and generated much acrimony and jealousy so that it became known as "the Helen of geometers" (after Helen of Troy). Among those who took up Mersenne's challenge were Gilles Personne de Roberval (1602-1675) and Evangelista Torricelli (1608-1647). By 1634 Roberval was able to prove Galileo's earlier hunch but he did not publish his proof. On the other hand Torricelli seems to have independently found two proofs in 1643 and he published these the next year. Torricelli did not state in his article that Roberval had proved the result earlier and Roberval wrote to Torricelli accusing him of plagiarism. It's amusing now to realize why Roberval did not publish his proof. Roberval was professor of mathematics in Paris at the Collège Royal (now Collège de France) and managed to hold this post for some forty years. Appointment to this position was determined every three years on the basis of a competitive examination, the questions of which were set by the incumbent. In 1634 Roberval won this contest probably because of the method he had developed to handle the cycloid. By not disclosing his method to others he successfully retained his position until his death but this meant he lost credit for most of his discoveries and he became embroiled in numerous quarrels with respect to priority.

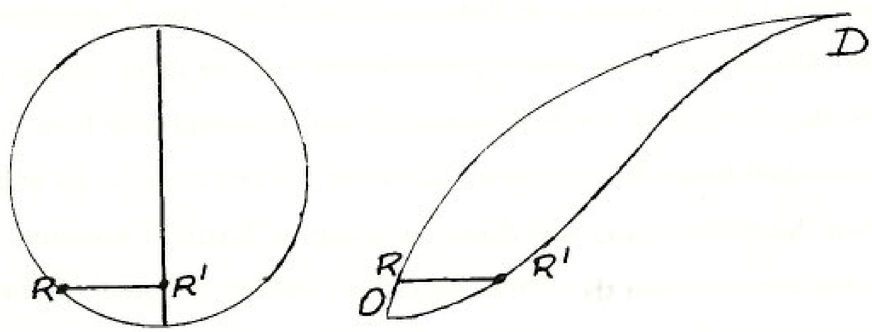
The key to Roberval's proof was to introduce a second curve called the **companion of the cycloid.**

Figure 2



Imagine the reflector initially at the bottom of the wheel at the origin 0 and imagine that the wheel completes one half a revolution when it rolls along OC so that the reflector eventually reaches D (see Figure 2). As the wheel rolls along OC the reflector moves to a generic position R where, since there is no slipping, $OA =$ arc AR. From R draw a line parallel to OC to meet the vertical diameter AB at R'. As R traces out the cycloid from 0 to D the point R' traces out a curve from 0 to D called the companion of the cycloid.

Figure 3



Roberval's insight was now to observe that the area between the cycloid and its companion is exactly one half the area of the wheel. Why is this the case? Consider the wheel in some fixed position and the cycloid and its companion as shown in Figure 3. Roberval, like other mathematicians of his period, thought of areas as sums of lines (of infinitesimal thickness). So the curved region $ORDR'O$ has area equal to the sum of all the lines RR' (as R moves from 0 to D). But, for a given point R on the cycloid the corresponding point R' is on the vertical diameter of the wheel. Thus we can shift RR' to the fixed wheel as shown in the figure. This means that the area of the semi-circle ARB is also the sum of the lines RR'. In other words the area between the cycloid and its companion is one half the area of the wheel or $\frac{1}{2}\pi r^2$.

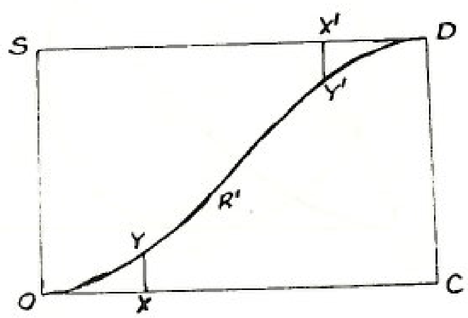


Figure 4

Next Roberval observed that the companion curve divides the rectangle $OCDS$ into two equal areas. To see this consider a general line XY perpendicular to OC at X which meets the companion curve at Y (see Figure 4). This line XY has the same length as the corresponding line $X'Y'$, drawn perpendicular to SD at X' where $DX' = OX$. (Prove this as an exercise.)

Therefore, area $OCDR'O$ = sum XY's = sum $X'Y'$'s = area $OR'DSO$.
This means area $OCDR'O = \frac{1}{2}$ area rectangle $OCDS$.
But $OC = \pi r = \frac{1}{2}$ circumference of wheel and $OS = 2r$ = diameter of wheel.

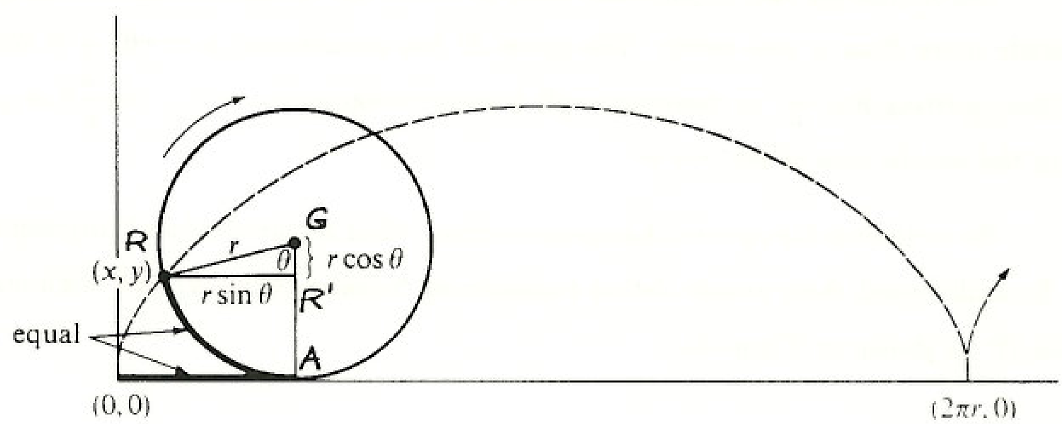
$$\text{Therefore, area } OCDR'O = \frac{1}{2}\pi r \times 2r = \pi r^2.$$

This means that the area under the cycloid (and above OC) is $\frac{1}{2}\pi r^2 + \pi r^2 = \frac{3}{2}\pi r^2$. Therefore the area between the (full arch of the) cycloid and its base is $3\pi r^2$ or three times the area of the generating wheel as divined by Galileo.

Certainly Roberval's proof reveals great insight and makes it very clear that the result is true. Nevertheless the proof is open to serious criticism. It does not make much sense to say that an area is a sum of lines since the area of a line presumable is zero (since it has zero width) and a sum of zeros is still zero.

How might we write down a modern proof of Galileo's inspired guess? (What we now do will make sense only if you have studied integration at school.)

Figure 5



Let the reflector R have coordinates (x, y). We know from integration theory that if we can solve for y in terms of x then the area is

$$\int_0^{2\pi r} y\ dx.$$

Rather than try to obtain y explicitly in terms of x we just find parametric equations for x and y. The parameter θ that we use is the angle between the radius vector RG and the perpendicular GA as defined in the diagram (see Figure 5). As the wheel rolls along in the positive direction θ increases from 0 (when the reflector R is at the origin), to π (when R is at the top of its arch) to 2π (when R has returned to the base line once more). We clearly have:

$$x = OA - RR' = \text{ arc } RA - RR' = r\theta - r\sin\theta = r(\theta - \sin\theta)$$

$$y = GA - GR' = r - r\cos\theta = r(1 - \cos\theta).$$

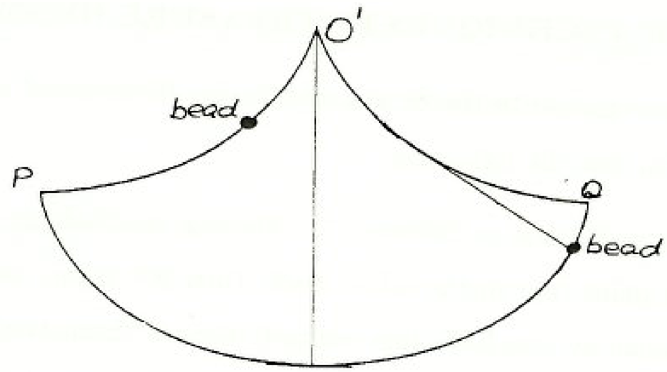
Hence the area is

$$\begin{aligned}\int_0^{2\pi r} y\ dx = \int_0^{2\pi} y(\theta)\frac{dx}{d\theta}d\theta &= \int_0^{2\pi} r(1-\cos\theta)r(1-\cos\theta)d\theta \\ &= r^2\int_0^{2\pi}(1-\cos\theta)r(1-\cos\theta)d\theta \\ &= r^2\int_0^{2\pi}(1-2\cos\theta+\cos^2\theta)d\theta = 3\pi r^2.\end{aligned}$$

Unfortunately this parametric description of the cycloid reveals its companion to be little more than a sine curve. The point R' has coordinates $x = r\theta$, $y = r(1-\cos\theta)$ so that, putting $\theta = \frac{x}{r}$, we have an explicit description as $y = r(1 - \cos\frac{x}{r})$ or $y = 1 - \cos x$ in the special case when $x = r$.

Nevertheless the cycloid does possess three other remarkable and surprising properties. To understand these results let us consider an "upside down cycloid' with cusp or vertex at O' as shown in Figure 6.

Figure 6



The first two of these properties were discovered by Christiaan Huygens (1629-1695) who was one of Galileo's immediate scientific successors.

The tautochrone property. Imagine the cycloid of Figure 6 as a smooth wire placed in a vertical plane and imagine a bead free to move under gravity on that wire (we disregard friction). Then the time of descent of the bead to the lowest point on the cycloid does not depend on the position from which the bead is released.

The isochrone property. Suppose that, instead of sliding on the cycloid, the bead is attached to the vertex O' by a light string constrained to move between the cycloidal arcs $O'P$ and $O'Q$. Then the period of oscillation of the pendulum (of given length) is independent of the amplitude of its swing. (Those readers doing 4 Unit Mathematics might be aware that this result is approximately true, for small amplitudes, for ordinary simple pendulums. Huygens tried to exploit his cycloidal pendulum to design an accurate mariner's chronometer which was crucial for measuring longitude. In 1735 John Harrison succeeded in making such a chronometer using a spring clock with a balance wheel.)

Huygen's discoveries of 1673 were followed in 1697 by an equally beautiful discovery made independently (and by different methods) by Johann Bernouilli (1667-1748) and his older brother Jakob (1654-1705).

The brachystochrone property. Consider a point A and a second lower point B not directly beneath A, and consider all curves from A to B down which a bead can slide under gravity. Then the curve which minimises the time taken to slide from A to B is the cycloid with A as vertex which passes through B.

We might consider these properties in more detail in a future article.