ADVANCED TECHNIQUES IN TREASURE HUNTING

One of the questions common to the Senior and Junior divisions of this year's School Mathematics Competition was the following:

> An old manuscript reads as follows: "...Having reached the island, walk from the palm tree to the white rock, turn 90° right, and walk the same distance as you have just walked (that is, from the tree to the rock). Place a peg in the ground. Return to the palm tree, walk to the black rock, turn 90° left, and walk a distance equal to that from the tree to the black rock. Place another peg in the ground. Dig for the treasure half way between the pegs." When you arrive, you find that the rocks are easily identifiable, but many more trees have grown up and it is impossible to tell which one was meant. Can you find the treasure?

A solution to this problem was given in the last issue of **Parabola**. A number of students, however, submitted excellent alternative solutions; three of these are presented in the following article.

First, perhaps I should mention some popular suggestions which *don't* work.

Some students proposed going over the whole island with a metal detector. Unfortunately the author of the old manuscript neglected to mention that the entire treasure consists of diamonds and credit cards (I guess the manuscript wasn't so old after all), so nothing will be found with a metal detector. Others suggested chopping down all the trees on the island and counting their rings to find out which is the original tree. Even more unfortunately, the island used to be a forestry research station devoted to breeding palm trees without rings. Then there were those who recommended bulldozing the whole island – most unfortunately of all, the treasure is protected by a large number of bombs buried throughout the island, so anyone attempting to use a bulldozer in the wrong place will find (or to be more exact, won't find) that their efforts come to a rapid conclusion.

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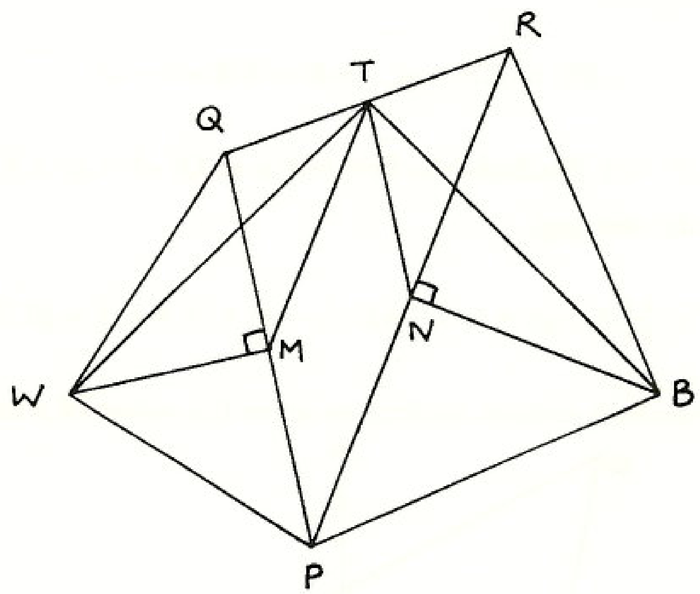
The most elegant solution submitted by any student used pure Euclidean methods. (However, "most elegant" doesn't always mean "most efficient" – for this see later.) Draw a diagram as shown below: W and B are the white and black rocks, P the palm tree, Q and R the first and second pegs respectively and T the treasure. Draw PQ and PR, and label their midpoints M and N; join TW, TB, TM and TN.



First consider $\triangle PRQ$. Since N and T are the midpoints of PR and QR we have $TN \| QP$, and similarly $TM \| RP$. Hence $TMPN$ is a parallelogram and we have $TN = MP$, $TM = NP$. From the given information, $\triangle PWQ$ and $\triangle PBR$ are right-angled isosceles triangles. Now in $\triangle TMW$ and $\triangle BNT$ we have

$$MW = MP = TN; \quad TM = NP = NB;$$

$$\angle TMW = 90^\circ + \angle TMQ = 90^\circ + \angle NPM = 90^\circ + \angle RNT = \angle BNT$$

and so these two triangles are congruent (SAS). Therefore $WT = TB$, that is, the treasure is equidistant from the two rocks. Moreover,

$$\begin{aligned}\angle WTB &= \angle WTM + \angle MTN + \angle NTB \\ &= \angle TBN + \angle RNT + \angle NTB \\ &= \angle TBN + \angle BNT - 90^\circ + \angle NTB \\ &= 90^\circ\end{aligned}$$

since the sum of the angles of $\triangle BNT$ is $180°$. Thus the treasure can be found by constructing a right-angled isosceles triangle on the hypotenuse WB.

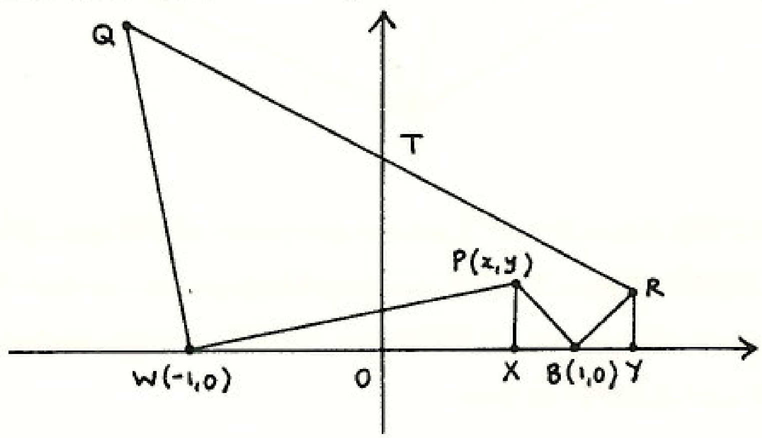
A second method of solving this problem is to use coordinate geometry. Set up a coordinate system with (say) the white rock at $W(-1,0)$ and the black rock at $B(1,0)$. Let the (unknown) position of the palm tree be $P(x,y)$. To find the coordinates of the second peg R we note that

$$BY = XP = y, \quad YR = XB = 1 - x$$

and so $R = (1+y,\ 1-x)$. Similarly the first peg is at $Q(-1-y,\ 1+x)$. The treasure is at the midpoint of the two pegs

$$T = \tfrac{1}{2}\Big((1+y) + (-1-y),\ (1-x) + (1+x)\Big) = (0,1).$$

Clearly this can be located without needing to know the values of x and y.



Probably the most efficient method is to work with complex numbers. Set up the coordinate system with (say) $W = -1$, $B = 1$ and $P = z$, an unknown complex number. Recalling that a rotation by a right angle anticlockwise corresponds to multiplication by i, we have

$$Q - W = i(P - W), \quad R - B = -i(P - B)$$

so that

$$Q = W + (Q - W) = -1 + i(z+1), \quad R = B + (R - B) = 1 - i(z-1);$$

hence

$$T = \tfrac{1}{2}(Q + R) = i,$$

which does not depend on the unknown z.

An interesting alternative method was also submitted for question 6 on the Senior paper:

> Twenty one 3×1 rectangles are placed without overlapping on a normal 8×8 chessboard, thus covering 63 of the 64 squares. Determine all possible locations of the uncovered square.

Label the rows of the chessboard A to H (top down) and suppose that the uncovered square is in the top row. We now consider in each row the number of squares covered by a 3×1 rectangle *placed vertically.* All the numbers I mention from here on are to be read "modulo 3", that is, "0" means "0 or 3 or 6 or ...", "1" means "1 or 4 or 7 or ..." and so on. Since a rectangle placed horizontally occupies 3 squares in the same row, the number of squares covered by a vertical rectangle is 1 in row A and 2 in every other row. To cover this 1 square in row A there must be 1 vertical rectangle covering rows ABC. Now one of the important squares in row B has been accounted for, so there must also be 1 vertical rectangle covering rows BCD. Row C now has all necessary squares filled in, so there are 0 rectangles covering rows CDE; similarly there must be 1 covering DEF, 1 covering EFG and 0 covering FGH. There is now no room to place another rectangle vertically; however in row G there is only 1 square covered by a vertical rectangle, whereas there should be 2. Therefore this case is impossible, and the uncovered square cannot be in row A. By a similar argument (try it!) you can show that the uncovered square is not in rows B or D. Finally, a symmetrical argument holds looking at the chessboard from any of its four sides; therefore the uncovered square can only be three rows in from each of the nearest edges, that is, in squares $(3,3), (3,6), (6,3)$ or $(6,6)$.