

therefore $9d \geq 2001$. Likewise, d has a multiple EOE which is not equal to d itself and is at most 898; so $2d \leq 898$. Hence

$$223 \leq d \leq 449;$$

but since d has the form OEO we can improve this to

$$301 \leq d \leq 389.$$

Now observe that in the last and second last subtractions, two different multiples of d were subtracted from two (possibly different) numbers EOE. But such a number is at most 898, which is less than $3d$, so the last two digits of q must be 12. This allows us to restrict the values of d still further. We have $602 \leq 2d \leq 778$; but $2d$ has digits EOE, so $610 \leq 2d \leq 698$ and

$$305 \leq d \leq 349.$$

To improve this once more we notice that the second subtraction is $\text{EOE} - d = \text{EO}$, where EO are the first two digits of $2d$; therefore

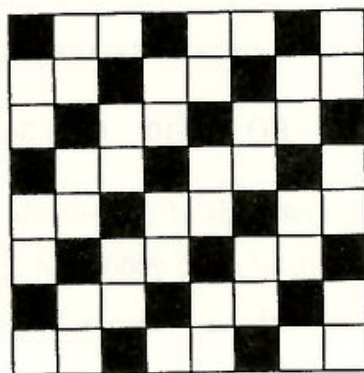
$$d = \text{EOE} - \text{EO} \geq 410 - 69 = 341.$$

Now we can determine the first digit of q : it is odd, and multiplied by d gives a result EEEO. But obviously $d, 3d, 5d < 2000$; and $3069 \leq 9d \leq 3141$; so to give a four-digit product beginning with an even number, the required digit must be 7. Hence $q = 712$ and we have five possibilities for d . If $d = 341$ or 343 then $2d$ is EEE, which is not so; if $d = 345$ then $7d$ is EEOO; while if $d = 349$ we find that dq is EEEEE, whereas it should be EEOEEE. Hence the only valid solution is given by $d = 347, q = 712$.

$$\begin{array}{r}
 \\
 347 \overline{) 247064} \\
 \underline{2429} \\
 416 \\
 \underline{347} \\
 694 \\
 \underline{694} \\
 0
 \end{array}$$

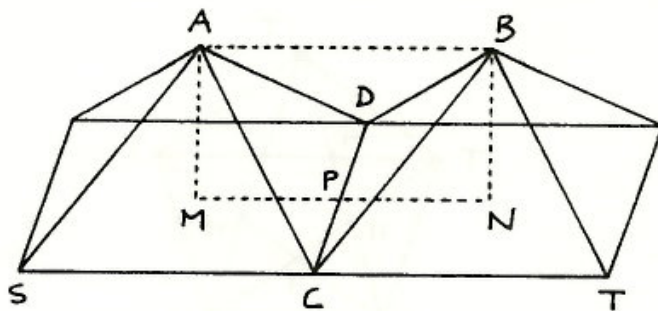
Q.895 Take a normal 8×8 chessboard and remove from it as few as possible individual squares, in such a way that no 3×1 rectangle can be placed so as to cover three squares on the remaining part of the board.

ANS. Label the chessboard with the numbers 1, 2, 3 as described in the Mathematics Competition solutions (Parabola issue 2, p.22). Then any 3×1 rectangle placed on the board must cover a 1, a 2 and a 3, and the figure 1 occurs twenty-one times. So by removing all the squares labelled 1 we make it impossible to place a 3×1 rectangle on the board. However, removing fewer than twenty-one squares will not do. To see this, consider the diagram on p.36 of the previous issue of Parabola. Here the board contains twenty-one 3×1 rectangles; so if only twenty or fewer squares are removed, at least one complete rectangle must remain. So the fewest possible number of squares that we may remove is twenty-one: one possibility is as shown below.



Q.896 (Adapted from a puzzle heard on ABC radio.) Take a square-based pyramid whose triangular faces are all equilateral, and a regular tetrahedron whose faces are of the same size as the triangular faces of the pyramid. Join these two solids along a pair of triangular faces. In the combined solid you will see two triangles which appear to lie very nearly in the same plane. Do they in fact lie exactly in the same plane? Prove your answer.

ANS. Yes, the two faces do lie in exactly the same plane. To see this, place *two* pyramids as shown in the diagram: the bases are coplanar and two edges meet.



Draw the interval AB joining the summits of the pyramids. Then it is clear that $ABCD$ is a tetrahedron with one of its faces attached to a face of (say) the left-hand pyramid; it is also clear that faces ASC and ABC (and BCT) are coplanar; what remains is to prove that the tetrahedron $ABCD$ is regular. But AC, AD, CD, BC and BD are sides of congruent equilateral triangles (given), while $AB = MN = 2MP$ and MP is half of DC , the side of the base of the pyramid. Hence $ABCD$ is a regular tetrahedron.

Q.897 Let $x = \sqrt[3]{10} + \sqrt[3]{6}$. Show that $x^3 - 3x\sqrt[3]{60} = 16$, and deduce (without a calculator!) that $x < 4$.

ANS. We have

$$\begin{aligned} x^3 &= (\sqrt[3]{10})^3 + 3(\sqrt[3]{10})^2(\sqrt[3]{6}) + 3(\sqrt[3]{10})(\sqrt[3]{6})^2 + (\sqrt[3]{6})^3 \\ &= 10 + 3(\sqrt[3]{10})(\sqrt[3]{6})(\sqrt[3]{10} + \sqrt[3]{6}) + 6 \\ &= 16 + 3x\sqrt[3]{60}, \end{aligned}$$

which proves the first part. If $x \geq 4$ then, noting that $\sqrt[3]{60} < \sqrt[3]{64} = 4$, we have

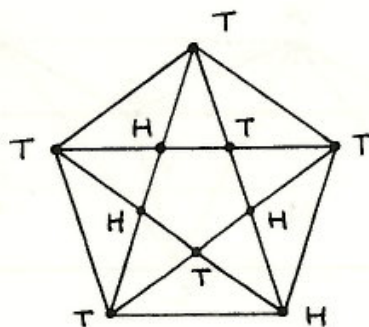
$$16 = x(x^2 - 3\sqrt[3]{60}) > 4 \times (16 - 12),$$

that is, $16 > 16$. This is obviously not true, so $x < 4$. (In fact a calculator gives $x = 3.9715\dots$)

Solved by: Lisa Gotley, All Saints Anglican School, Merrimac, Qld.

B. David, North Bondi, NSW.

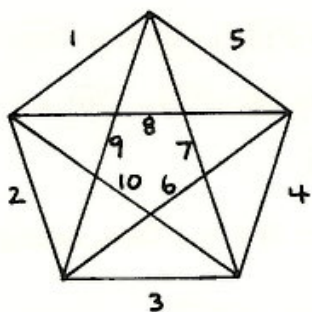
- Q.898 A pentagon with all its diagonals drawn in has a coin placed on each intersection of lines, with either heads or tails facing upwards (see the diagram).



It is permitted to select any one of the ten lines in the figure, and turn over all the coins lying on that line; such a move may be performed as often as you like.

- (i) Find a sequence of moves starting with the above position and finishing with all coins facing up heads.
- (ii) Find a rule by which it is possible to tell merely by studying any initial position whether or not the task in (i) can be accomplished.

- ANS. (i) This can be achieved by turning over the coins on lines 2, 3, 6 and 7 (see diagram). The lines can be treated in any order, but there is no solution in fewer than four moves, and no other solution (I think) in just four.



- (ii) Note that any allowable move flips two coins in the outer "ring" of five. Thus any move increases or decreases the number of tails among these coins by two, or (if the coins affected were originally a head and a tail) leaves it unchanged. Since our aim is to reduce the number of tails to zero, we must start with an even number of tails in the outer ring. Likewise, any move affects either two coins in the inner ring or none, and we must start also

with an even number of tails in the inner ring if we are to have any chance of success.

If the above conditions are met, does this guarantee that the problem is solvable? Yes, for we can obey the following procedure. If either of the "inner" coins on line 6 is tails we flip line 6; then if either "inner" coin on line 7 is tails we flip line 7; and similarly for lines 8 and 9. We then have four heads on the inner ring, and since we started with an even number of tails and changed none or two every move, the fifth coin must be heads too. We can now finish the game by treating lines 1, 2, 3, 4 in a similar way.

- Q.899** (i) We have seven coins, apparently identical, of which two are heavier than the other five (and the two heavy coins weigh the same as each other). With three weighings on a beam balance, find the heavy coins.
- (ii) Show that the above problem cannot be solved if we have eight coins, with two heavier than the other six.

- ANS.** (i) Label the coins 1, 2, ..., 7. One possible method is the following:
- (1) weigh 1,2 against 3,4;
 - (2) weigh 1,4 against 5,6;
 - (3) weigh 1,3,5 against 2,4,6.

We can now work out the results of the weighings for a given pair of heavy coins. If, for example, coins 1 and 5 are heavy, then the first weighing tips to the left, the second balances, and the third again tips to the left. Considering in this way all possible pairs, we find that each gives a different set of results for the three weighings, and so these results will determine which two coins are in fact the heavy ones.

- (ii) Each weighing will give one of three results (left, right or balance). Therefore the total number of results that can arise from a sequence of three weighings is $3 \times 3 \times 3 = 27$. If we have eight coins there are $\frac{1}{2} (8 \times 7)$, that is, 28 possibilities for the heavy pair. Hence, if we weigh the coins by any procedure whatsoever, there must be two possibilities which give the same result and

therefore cannot be distinguished.

Q.900 Consider the binomial expansion of $(x + 1)^n$.

- (i) If the expansion contains three consecutive coefficients such that the second and third are (respectively) twice and three times the first, find n .
- (ii) If it contains three consecutive coefficients such that the second and third are respectively a times and 23 times the first, where a is an integer, find a and n .

ANS. (i) We have

$$\binom{n}{j+1} = 2 \binom{n}{j} \quad \text{and} \quad \binom{n}{j+2} = 3 \binom{n}{j} = \frac{3}{2} \binom{n}{j+1},$$

where $\binom{n}{j}$ is the first of the three binomial coefficients. Using the formula

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

and doing a lot of cancellation, we find

$$n - j = 2(j + 1), \quad 2(n - j - 1) = 3(j + 2).$$

Simplifying,

$$n = 3j + 2, \quad 2n = 5j + 8.$$

Finally, taking three times the second equation minus five times the first yields the solution $n = 14$. We can check this result by noting that $j = 4$ and the three binomial coefficients are

$$\binom{14}{4} = 1001, \quad \binom{14}{5} = 2002, \quad \binom{14}{6} = 3003.$$

(ii) In the same way we get

$$n = (a + 1)j + a, \quad an = (a + 23)j + (a + 46).$$

We can solve for n as above, but the algebra is a little easier if we solve for j instead. Taking a times the first equation minus the second and rearranging leads to

$$j = -\frac{a^2 - a - 46}{a^2 - 23}.$$

Note that $j > 0$, so one of the expressions $a^2 - a - 46$ and $a^2 - 23$ must be positive, the other negative. But clearly $a^2 - 23$ is the larger, so we have

$$a^2 - 23 > 0, \quad a^2 - a - 46 < 0$$

and hence, remembering that a is an integer, $5 \leq a \leq 7$. Of these, only $a = 5$ gives an integer value for j , namely $j = 13$; we then find $n = 83$. If you are willing to do a little (?) arithmetic you can check that

$$\begin{aligned} \binom{83}{13} &= 528955739755020, \\ \binom{83}{14} &= 2644778698775100, \\ \binom{83}{15} &= 12165982014365460 \end{aligned}$$

and that the second and third of these are in fact 5 and 23 times the first.

Solved by: B. David, North Bondi, NSW.

- Q.901** In the mythical country of Ozz there are three kinds of coins: in order of increasing value the cong, the dong and the fong. Two Ozzians, Alice and Bert, each with his or her own pile of congs, donges and fongs, play the following game. Alice chooses any one of her coins, whereupon Bert must take from his pile one each of the other two kinds. They then toss the three coins and add up the value of those which fall heads; the player whose total is greater takes all three coins. There are no ties except when all three coins turn up tails, in which case each player keeps his or her coin(s). Alice and Bert notice that in the long run neither player has any advantage, regardless of which coin Alice selects. How many congs are in a fong?

ANS. Let the value of a cong, a dong and a fong be c, d and f respectively, and note that $c < d < f$. Consider the case where Alice chooses a cong. There are eight ways in which the coins can fall. If they all fall tails, the game is drawn and neither player wins anything. There is one case (the cong falls heads, the other two tails) in which Alice wins $d + f$, and six cases (all the others) in which Bert wins c . So on average, in any seven games, Alice wins $d + f$ and Bert wins $6c$. But since neither player has an advantage these amounts must be equal: that is, $d + f = 6c$, or

$$6c - d - f = 0. \quad (1)$$

If Alice chooses a dong we can analyse the game similarly (she has two ways of winning: the dong must fall heads and the fong tails, but the cong may be either) to obtain the equation

$$2c - 5d + 2f = 0. \quad (2)$$

Now suppose Alice chooses a fong. Since no tie is possible other than "all tails", f cannot equal $c + d$. We must consider the cases $f < c + d$ and $f > c + d$. In the former, Alice may win in any of three ways: the fong must fall heads, and the cong and dong must be tails and tails, heads and tails or tails and heads respectively. This gives the equation

$$3c + 3d - 4f = 0. \quad (3a)$$

However (1), (2) and (3a) have no solution except for $c = d = f = 0$, which clearly is not a sensible answer to the present problem. If $f > c + d$, then Alice wins in four cases and we have

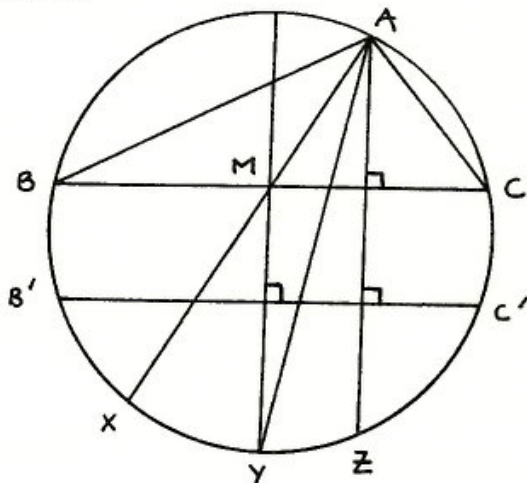
$$4c + 4d - 3f = 0. \quad (3b)$$

Solving (1), (2) and (3b) we find that c may have any value, and $f = 4c$. So a fong is worth 4 congs. (Also a dong is worth 2 congs. So if we decide to take $c = 5$ or $c = 50$, the country of Ozz need not be entirely mythical after all.)

Solved by: B. David, North Bondi, NSW.

Q.902 Given three non-collinear points X, Y, Z such that $\angle XYZ$ is obtuse, show how to construct a triangle $\triangle ABC$ such that the median of the triangle through A intersects the circumcircle at X , the angle bisector through A intersects the circumcircle at Y , and the altitude through A intersects the circumcircle at Z .

ANS. Firstly, since the points X, Y, Z are given we can draw through them the circumcircle of $\triangle ABC$. Now since AY bisects the angle at A we have equal angles $\angle BAY$ and $\angle CAZ$, which



are subtended by equal arcs BY, CY . Draw equal arcs (of any length) each side of Y , cutting the circle in B' and C' ; then the line $B'C'$ is parallel to the unknown line BC . From the given information AZ is perpendicular to BC , therefore also perpendicular to $B'C'$. Thus we may draw a line through Z , perpendicular to the known line $B'C'$, and A is the other point of intersection of this line with the circumcircle. Finally, we need to find the side BC , which is bisected by AX . Draw AX , draw a diameter of the circumcircle perpendicular to $B'C'$, call their intersection M , and draw a line through M parallel to $B'C'$. Then M is the midpoint of the line (a diameter of a circle bisects any chord perpendicular to it) and hence AX bisects the line, which is therefore the required BC . This completes the construction.

Late solutions to problems from issue 1, 1993:

Jonathan Kong (Sydney Grammar School) sent in solutions to questions 882, 883 and 884.