Parabola Volume 30, Issue 1 (1994)

ADDITION GAMES

David Angell*

We'll look at a few two-player games, each of which can be broadly described as follows: the first player announces a number; the second adds to it another number, chosen subject to certain rules, and announces the sum; the first adds yet another number and announces the new sum; and so on alternately. The winner will be the player who first reaches a specified total which we'll call the "target number", or who forces his opponent above that total. Throughout this article, "number" will mean "positive integer". For some of the games we examine it will be possible to give a winning strategy for one player or the other: that is, a set of rules which that player can follow in order to guarantee himself a win, regardless of what his opponent may do. Indeed, finding such a strategy will always be possible in principle, though sometimes the necessary calculations would be too hard to accomplish in practice.

Our first game will be defined simply by restricting the numbers added at each step to $1, 2, \ldots, 9$ and specifying the target number 100. A possible game would be

$$8, 17; 26, 31; 32, 41; 50, 53; 61, 62; 71, 76; 84, 90; 91, 100.$$

These numbers are the accumulated totals announced by the two players; to make it easier to read, the first player's totals are followed by commas, the second by semi-colons. Note that the number added at each step is indeed from 1 to 9, according to the rules of the game. It's fairly clear that in this example both players have played rather poorly: for example, if the first player had announced 80 instead of 84, then the second would have made the total somewhere from 81 to 89; the first could have increased it to 90, thus enabling himself on his next move to reach 100 and win the game. Indeed, this shows us how to devise a winning strategy. Whenever your opponent adds n, you can add $10-n$, so that the total has increased by 10 since your last move. Thus you can be sure of reaching 100 if you can reach 90. Likewise, you can be sure of reaching 90 provided you can reach

* David is a mathematics tutor at UNSW. His last article for *Parabola* contained no equations either.

$80, 70, \ldots, 10$. Thus the second player can force a win by ensuring that the total is always a multiple of 10. On the other hand, if he ever accidentally makes the total anything other than a multiple of 10 the first player can reach the next multiple of 10 and then win by following the same strategy.

If you play this game with someone who doesn't know the winning strategy, they will presumably notice that you always make the total a multiple of 10, and catch on fairly quickly. A variation making this a little less obvious is to allow the numbers $1, 2, \ldots, 10$ to be added. Then you can always answer n with $11 - n$, and so the key totals are $100, 89, 78, \ldots, 12, 1$. In this version of the game, therefore, the first player can force a win by starting with 1; otherwise the second player will always be able to win. You might like to find out who has the advantage if the numbers to be added are $1, 2, \ldots, m$ and the target number is t.

A minor adjustment to the above rules makes the game much harder to play carefully. Once again, the numbers from 1 to 9 may be added; except that in this version neither player may add the same number as his opponent has just used. To keep things relatively simple let's take 30 as the target number. Why not find an opponent and play this game a few times before reading further?

We can analyse the game much as we did the previous one by starting at the goal and working backwards. We'll concentrate on finding the moves which, starting at some particular total, guarantee a win. If the total is 30, obviously any move loses; if the total is 29, adding 1 makes it 30 and wins, while adding more than 1 loses. If the total is 28 it is clear that 2 wins and anything bigger loses; but 1 is also a winning move since it makes the total 29 and bars the opponent from the only possible winning reply. If the total is 27 then 1 loses since the opponent can reply 2; 2 loses since the opponent can reply 1; 3 wins; and anything larger loses. We can continue in this way to construct a table of winning moves. A little thought will show that if the total is n, then adding k leads to a win if the following conditions are met:

(i) $n + k$ does not exceed the target number;

(ii) either there are no winning moves from total $n + k$, or k itself is the only winning move.

The complete table is

total	moves	total	moves	total	moves	total	moves
30	none	20	5	10	9	0	8
29	1	19	none	9	5		
28	1, 2	18	1	8	none		
27	3	17	1, 2	7	1, 6		
26	4	16	3, 7	6	2		
25	5	15	4, 5	5	3		
24	3, 6	14	5	4	2, 4, 5		
23	7	13	6	3	5, 8		
22	4, 8	12	7, 9	2	3, 6		
21	9	11	8	1	7, 9		

and hence the first player can force a win by starting with 8.

Though this gives a complete strategy for playing the game, it is not a very satisfactory one as you could hardly memorise the whole table – certainly not if the target number were larger. What we would like is a simple, easily remembered way of finding a winning move (if there is one) in every situation. We can find a useful, though not complete, method by looking at the totals where no winning move exists – in the above example 30, 19 and 8. If we can reach one of these totals, which I shall call the "key totals", our opponent cannot win. There are two questions we need to answer:

(i) before playing the game, how can we determine the key totals efficiently (without calculating the whole table)?

(ii) once our opponent has moved away from one key total, how do we reach another (again, without recourse to a table)?

By looking at the results when the target number is 30 you may be able to guess an easy way of finding the key totals – unfortunately your guess will probably be wrong as the pattern changes for higher target numbers! If the target is 100 the key totals are in fact

$$100, 89, 78, 68, 57, 46, 36, 25, 14, 4.$$

See if you can guess the pattern from this information and then prove that your guess is always correct. (Hint: nothing particularly clever is involved – if you draw up the table for target 50 and study it carefully you should be able to find the reason behind the pattern.)

This (assuming you have guessed correctly!) answers question (i). Question (ii) is harder to answer precisely. Remember that in the very first game we discussed the second player could force a win, moving step by step through the key totals $0, 10, \ldots, 100$. But here it is not so simple. Though you reach one key total, the opponent may be able to prevent you from taking the next: you then have to skip it and aim for some future key. This is where my "method" is incomplete. Rather than trying to develop a fully detailed system I shall give a sample game. Here the target is 100, and so the keys are as shown above. The letters refer to comments following the game.

$$4a, 9b; 17c, 21; 26d, 27; 36, 39; 46, 47; 52e, 60; 64, 71f; 80g, 84; 89, 90; 95h.$$

Comments. (a) Player 1 reaches the first key total ... (b) ... but is prevented from reaching the next, 14. (c) Stops player 2 reaching 25. Adding 6 to give a total of 15 would also do this, but would lose in the long run. (d) Now player 2 cannot add 5, so player 1 will reach 36 next move. (e) The only way to prevent 57. (f) Player 2 sees that he cannot reach 68, so skips it and blocks 78. (g) Now player 2 cannot add 9, so player 1 will reach 89 next move. (h) Since player 2 cannot play 5 it is now clear that he must lose.

To conclude, I'll just describe two more games and leave you to experiment with them. The first is not too hard to investigate using the above methods, but the second is probably best learnt by experience.

In the first game the number added by each player must be more than that added by the previous player, but no more than twice as much. Thus, for example, if one player adds 6, the other must then add 7, 8, 9, 10, 11 or 12. Clearly some restriction is necessary on the initial move, otherwise the first player forces an immediate win by choosing half the target number or more. I suggest that a reasonably interesting game can be played with target 100 and initial move restricted to 5 or less. See if you can determine what moves (if any) guarantee a win for the first player.

Our final game is based on "Arithmetical Croquet", invented by Lewis Carroll, author of "Alice in Wonderland". In real life Carroll was Charles Lutwidge Dodgson, lecturer in mathematics from 1855 to 1881 at Christ Church College, Oxford. The rules of Carroll's game are somewhat difficult to understand unless you are familiar with the laws and the aims of croquet: though an essential social accomplishment in the nineteenth century, this is less so nowadays, and I have tried to work out what Carroll meant and recast the rules into a croquet-free form. The original game can be found in "The Magic of Lewis Carroll", edited by John Fisher, Penguin, 1981. Note that here each player keeps a separate total, rather than adding a number to the opponent's previous total.

1. At each move the player may add any number from 1 to 8 (but see rule 3).
2. The winner is the first player to reach 100 after "taking" the numbers $10, 20, \ldots, 90$. There are two ways of "taking" a number.
   (a) Go from below the number to the same distance above, e.g., going from 17 to 23 "takes" 20. Going to any other number above 20 means that 20 has been "missed". In this case the player must on his next move *subtract* some number from his total in order to go back below 20 and then take it properly. If a player "misses" the same number twice he loses.
   (b) Play to the required number on one move, and then add the same number next move, e.g., 17 to 20 and then 20 to 23. If the required "exit number" is impossible (see rule 3 for how this may be) the player loses a turn.
3. The number added may not be the same as the opponent has just added, nor 9 minus this number (for example, 2 may not be met by 2 or 7), except
   (a) if the opponent's total is $10, 20, \ldots$ or 90;
   (b) if the opponent's total is more than 90 (unless the player's own total is also more than 90).

   For example, if one player goes from 48 to 52, the opponent may play any number except 4 or 5; if the first plays from 48 to 50, the second may play any number at all (say 2 or 7, thus forcing the first to miss a turn – see rule 2(b)).
4. If a player's total exceeds 100, conditions similar to rule 2(a) apply: that is, the player must go back below 100 and try again. "Missing" 100 twice loses the game.
5. If one player's total is $10, 20, \ldots$ or 90 the other player may not add the same number twice consecutively. For example, if one player goes from 17 to 20, the other may not add $3, 3, 3 \ldots$ or $6, 6, 6, \ldots$; but may keep the first player at 20 by adding $3, 6, 3, 6, \ldots$ for as long as this is legal.