PICK'S THEOREM, INEQUATIONS AND MAGIC SQUARES

Peter Brown

Mathematics can at times appear fairly dull and routine. Solving 30 quadratic equations, finding the hypotenuse of seemingly endless right triangles and differentiating even more functions can be downright boring!

For me, mathematics becomes really fascinating and worthwhile when different ideas and techniques come together in problem solving.

I want to show you a fairly simple yet interesting problem whose solution involves basic ideas from equations, co-ordinate geometry, inequations and Pick's Theorem (for finding areas).

Suppose we have a square block of nine positive whole numbers, such that the sum of any row, column or diagonal is some fixed constant k, e.g.,

$$\begin{pmatrix} 6 & 3 & 3 \\ 1 & 4 & 7 \\ 5 & 5 & 2 \end{pmatrix}, \quad \text{where } k = 12.$$

I will call such a block, a *pseudo-magic square*, since the numbers do not have to be distinct.

The two questions I wish to ask are:

(i) Given a positive integer k is there a pseudo-magic square?

(ii) If so, how many such squares are there?

The general square has the form

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

and we require that the following eight equations be satisfied

$$\begin{aligned} a+b+c&=k \qquad & b+e+h&=k \\ d+e+f&=k \qquad & c+f+i&=k \\ g+h+i&=k \qquad & a+e+i&=k \\ a+d+g&=k \qquad & c+e+g&=k. \end{aligned}$$

These simultaneous equations can be solved by setting $b = r, c = s$ (r, s positive whole numbers) and then

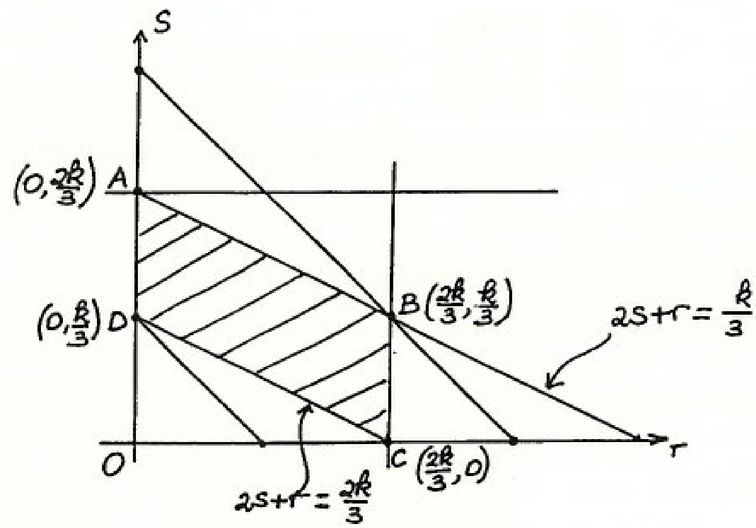
$$\begin{array}{ll} a = k - r - s & g = \frac{2k}{3} - s \\ d = 2s - \frac{2k}{3} + r & h = \frac{2k}{3} - r \\ e = \frac{k}{3} & i = r + s - \frac{k}{3}. \\ f = \frac{4}{3}k - 2s - r & \end{array}$$

You should try and obtain these yourself or at least check they work.

We see then, that k must be a multiple of 3 which answers the first question. Now, since all the entries are positive, we obtain a set of inequations:

$$\begin{array}{ll} r > 0 & s < \frac{2k}{3} \\ s > 0 & r < \frac{2k}{3} \\ r + s < k & r + s > \frac{k}{3} \\ 2s + r > \frac{2k}{3} & 2s + r < \frac{4k}{3} \end{array}$$

These inequations appear rather daunting **until** we graph them!



The region which obeys **all** the inequalities is defined by

$$0 < r < \frac{2k}{3},\ \frac{2k}{3} < 2s + r < \frac{4k}{3}$$

and is shaded in the diagram.

To solve the second question, we want to count the number of ordered pairs (r, s), with r and s positive whole numbers, which lie completely **inside** the parallelogram. Such points are called *lattice points*. To achieve this I will use Pick's Theorem, which is usually taught in Year 8 at school. It says that the area A of a plane figure bounded by straight lines and with lattice points as vertices is given by

$$A = \frac{1}{2}B + I - 1,$$

where B is the number of lattice points on the boundary and I is the number of lattice points inside the figure. So, if we can find A and B then we can determine I. Since k is a multiple of 3, we let $k = 3j$ and on AD there are $\frac{k}{3} + 1 = j + 1$ lattice points and then similarly there are $j + 1$ points on BC.

Now the equation of DC is

$$2s + r = \frac{2k}{3}$$

or $s + \frac{r}{2} = j$, hence r must be even.

So on DC, the number of lattice points is $j+1$ since r can take the values $0, 2, 4, \cdots, 2j$. Clearly the same thing happens on AB. Noticing that we have counted each of the corner points **twice**, the total number of lattice points on the boundary is

$$4(j + 1) - 4 = 4j.$$

Also the area of the parallelogram is $2j^2$ and so

$$\begin{aligned} I = A - \frac{1}{2}B + 1 &= 2j^2 - \frac{1}{2}(4j) + 1 \\ &= \frac{2k^2}{9} - \frac{2k}{3} + 1. \end{aligned}$$

Hence, for any given positive k, (a multiple of 3) there are $\frac{2k^2}{9} - \frac{2k}{3} + 1$ pseudo- magic squares with sum k. For example, if $k = 6$, there are 5 such squares, with

$$(r, s) = (1,3),\ (3,2),\ (2,2),\ (3,1),\ (1,2)$$

$$\begin{pmatrix} 2 & 1 & 3 \\ 3 & 2 & 1 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 3 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}.$$

Try finding the 13 squares for $k = 9$, systematically, by drawing up the parallelogram and finding the lattice points. Remembering that $b = r, c = s$ and $e = \frac{k}{3}$, you should be able to find these squares very quickly.