SOLUTIONS TO PROBLEMS 903-912

Q.903 Find all positive integers n such that $1+2+3+\cdots+n$ is a factor of $1\times2\times3\times\cdots\times n$.

ANS. We use the notation $a \mid b$ (read "a divides b") to mean that a is a factor of b. Summing the arithmetic progression, we want

$$\frac{1}{2}n(n+1) \mid 1 \times 2 \times 3 \times \ldots \times n,$$

that is,

$$n+1 \mid 2 \times 1 \times 2 \times 3 \times \ldots \times (n-1).$$

If $n+1$ is prime it must divide some factor on the RHS. Except for the case $n+1=2$ this is impossible as all factors on the RHS are less than $n+1$. If $n+1$ is composite, write $n+1=ab$ with $a,b>1$. Then

$$a, b \leq \frac{n+1}{2} < n,$$

so both a and b occur on the RHS. If $a \neq b$ this shows $n+1 \mid \text{RHS}$; if $a=b \geq 3$ then a and $2a$ occur on the RHS, so $n+1 \mid \text{RHS}$; if $a=b=2$ then the statement is $4 \mid 2\times1\times2$ which is true. To sum up, $1+2+3+\ldots+n$ is a factor of $1\times2\times3\times\ldots n$ if and only if n is not one less than an odd prime.

Solved by: Lisa Gotley, All Saints' Anglican School, Merrimac, Qld.

Q.904 The average value of the coins in my pocket is 83 cents. After taking out two coins, not both the same, I find that the average has increased to 90 cents. What were the two coins? (All current Australian coins are allowable, that is, 5c, 10c, 20c, 50c, \$1 and \$2.)

ANS. Solution by Belinda Gotley, All Saints' Anglican School, Merrimac, Qld. (slightly rephrased).

Let the values in cents of the two coins be x and y. Removing these coins, we can imagine their remaining "share" of the average, $(83-x)+(83-y)$, to be distributed among the other coins. Since this boosts the average by 7, we see that $166-(x+y)$ is divisible by 7. Since $x+y$ is also a multiple of 5, we have

$$x+y=5, 40, 75, 100, 145.$$

And the only one of these that works is $x+y=110$ with $x=100$ and $y=10$. So the coins are a \$1 coin and a 10c coin.

Also solved by: Lisa Gotley, All Saints' Anglican School, Merrimac, Qld. (independently).

Katy Lai, James Ruse AHS.

Paul Shepherd, Smiths Hill HS, Wollongong.

Q.905 Find all solutions of

$$x^4 - 4x^3 - 2x^2 + 12x + 8 = 0.$$

ANS. Paul Shepherd of Smiths Hill HS, Wollongong, noticed that we have

$$\begin{aligned} x^4 - 4x^3 - 2x^2 + 12x + 8 &= x^4 - 4x^3 + 6x^2 - 4x + 1 \\ &\qquad - 7x^2 + 16x + 7 \\ &= (x-1)^4 - 8(x^2 - 2x + 1) + 15 \\ &= (x-1)^4 - 8(x-1)^2 + 15. \end{aligned}$$

So, substituting $y = (x-1)^2$ we have

$$y^2 - 8y + 15 = 0$$

which gives $y = 5$ or $y = 3$. Hence we have two quadratics

$$(x-1)^2 = 5, \; (x-1)^2 = 3$$

and the four solutions are

$$x = 1 \pm \sqrt{5}, \; 1 \pm \sqrt{3}.$$

An alternative method (Lisa Gotley, All Saints' Anglican School, Merrimac, Qld.) is to write down the factorisation

$$x^4 - 4x^3 - 2x^2 + 12x + 8 = (x^2 + ax + b)(x^2 + cx + d).$$

Expanding the RHS and equating coefficients,

$$a + c = -4,\ d + b + ac = -2,\ ad + bc = 12,\ bd = 8;$$

and with a bit of work the solution

$$a = -2,\ b = -2,\ c = -2,\ d = -4$$

can be found. Thus

$$(x^2 - 2x - 4)(x^2 - 2x - 2) = 0$$

and we obtain the same solutiosn as above.

***Q.906** (a) All of the following.

(b) None of the following.

(c) Some of the following.

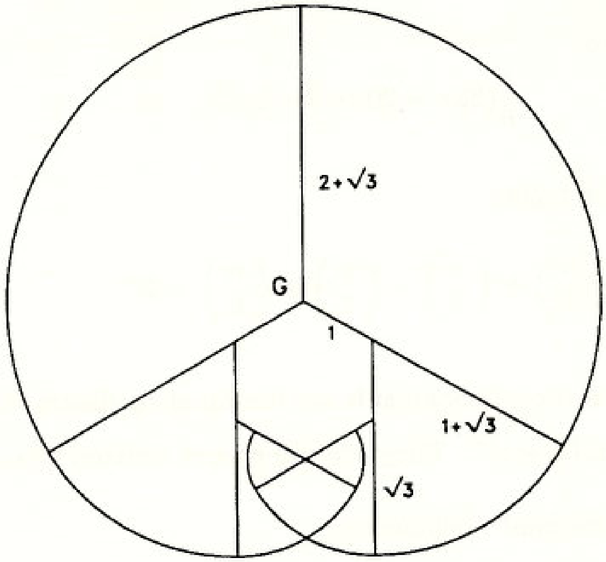
(d) All of the above.

(e) None of the above.

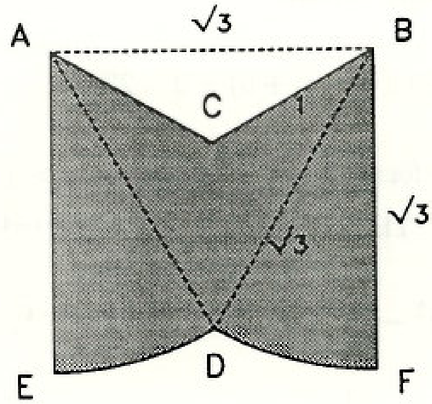
ANS. This is a multiple-choice question with no information given! Nevertheless it is possible to deduce the answer. If (d) is true then (b) is true and so (d) is false after all. Since (d) is false, (a) is false. If (c) is true then (e) is true (since we know (d) is not) and hence (c) is false. Since (c) is false, all of those following are false: that is, (e) is false. Hence (b) is true. Now we have to check that all the true and false specifications are consistent (otherwise there will be no solution to the problem). This is easily done, and so the answer is (b).

Q.907 In the middle of a large field a patch is fenced off in the shape of a regular hexagon. A goat outside the hexagon is tethered by a rope attached to one corner of the hexagon and having length $2 + \sqrt{3}$ times the side of the hexagon. Find the total area within which the goat can graze.

ANS. The following method of solution is by Lisa Gotley, All Saints' Anglican School, Merrimac, Qld. Noting that $3 < 2 + \sqrt{3} < 4$, the grazing area looks like this,



where G is the point to which the goat is tied and we have chosen the hexagon to have side length 1. The reachable region consists of two thirds of a circle with radius $2 + \sqrt{3}$, two sixths of a circle with radius $1 + \sqrt{3}$, and the shaded area below which we examine more closely.



We can calculate the area as that of two twelfths of a circle ADE and BDF, plus an equilaterial triangle ABD, minus the triangle ABC. Using well-known formulae, the total area within which the goat can graze is

$$\frac{2}{3}\pi(2+\sqrt{3})^2 + \frac{2}{6}\pi(1+\sqrt{3})^2 + \frac{2}{12}\pi(\sqrt{3})^2 + \frac{3\sqrt{3}}{4} - \frac{\sqrt{3}}{4}$$

which simplifies to

$$\frac{1}{6}(39\pi + 20\pi\sqrt{3} + 3\sqrt{3}).$$

Q.908 Solve in positive integers

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} = 2^m.$$

(Here the terms on the left hand side are binomial coeffients, that is, $\binom{n}{k}$ is the coefficient of x^k in $(1+x)^n$. They are sometimes written $C(n,k)$ or ${}^nC_k$.)

ANS. Expanding the binomial coefficients,

$$\begin{aligned} LHS &= 1 + n + \frac{1}{2}n(n-1) + \frac{1}{6}n(n-1)(n-2) \\ &= \frac{1}{6}(n^3 + 5n + 6) \\ &= \frac{1}{6}(n+1)(n^2 - n + 6) \end{aligned}$$

and so the equation can be written

$$(n+1)(n^2 - n + 6) = 3 \times 2^{m+1}.$$

Therefore $n+1$ has the form 2^k or 3×2^k, where $k \geq 0$.

Case 1, $n + 1 = 2^k$. Then $n^2 - n + 6 = 3 \times 2^{m+1-k}$ and we have

$$(2^k - 1)^2 - (2^k - 1) + 6 = 3 \times 2^{m+1-k}.$$

Simplifying,

$$2^{2k} - 3 \times 2^k + 8 = 3 \times 2^{m+1-k}. \qquad (*)$$

Now if $k \geq 4$ then the LHS can be written

$$(2^{2k-3} - 3 \times 2^{k-3} + 1) \times 2^3$$

as the product of an odd number and a power of 2. Comparing with $(*)$ we must have $m + 1 - k = 3$. Substituting into the quadratic above gives $n^2 - n + 6 = 24$, which has no solutions in integers. Looking at the cases where $k < 4$ individually, we have

$$\begin{array}{lllllll} k=3 & \Rightarrow & n=2^3-1=7 & \Rightarrow & 48=3\times 2^{m-2} & \Rightarrow & m=6 \\ k=2 & \Rightarrow & n=2^2-1=3 & \Rightarrow & 12=3\times 2^{m-1} & \Rightarrow & m=3 \\ k=1 & \Rightarrow & n=2^1-1=1 & \Rightarrow & 6=3\times 2^{m} & \Rightarrow & m=1. \end{array}$$

Case 2, $n+1 = 3\times 2^k$. Then $n^2-n+6 = 2^{m+1-k}$ and as above we find for $k \geq 4$

$$(9\times 2^{2k-3} - 9\times 2^{k-3} + 1)\times 2^3 = -2^{m+1-k}.$$

Therefore $m+1-k = 3$ again and $n^2-n+6 = 8$, which has $n = 2$ as the only positive integer solution. If $k < 4$ we have

$$\begin{array}{lllllll} k=3 & \Rightarrow & n=3\times 2^3-1=23 & \Rightarrow & 512=2^{m-2} & \Rightarrow & m=11 \\ k=2 & \Rightarrow & n=3\times 2^2-1=11 & \Rightarrow & 106-2^{m-1} & & \\ k=1 & \Rightarrow & n=3\times 2^1-1=\ \ 5 & \Rightarrow & 26=2^{m} & & \end{array}$$

and the last two do not give integer values for m. Hence the solutions are

$$n=1, m=1;\ n=2, m=2;\ n=3, m=3;\ n=7, m=6;\ n=23, m=11.$$

[*Note.* You may have seen $\binom{n}{k}$ defined only for $0 \leq k \leq n$. In fact it is possible to define the symbol for all integers $k \geq 0$; if n is an integer and $k > n \geq 0$ then $\binom{n}{k} = 0$.]

Partial solution: Paul Shepherd, Smiths Hill HS, Wollongong.

***Q.909** Is it possible to place the numbers $1, 2, 3, \ldots, 10$ around a circle in such a way that the maximum sum of three adjacent numbers is (i) 16; (ii) 17; (iii) 18? If so, show how to do it; if not, explain why not.

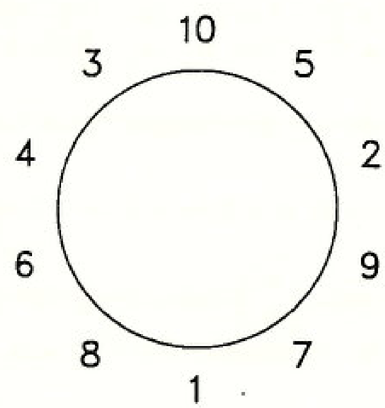
ANS. Solution by Paul Shepherd, Smiths Hill HS, Wollongong.

(i) If 16 is the maximum sum of any three adjacent numbers, then the sum of the numbers around the circle cannot exceed $\frac{10\times 16}{3} = 53\frac{1}{3}$, since we count each number three times by considering the sum of these maximums around the circle. This is impossible since their sum is 55.

(ii) The case for 17 is also impossible. To show this we show that if the sum of three consecutive numbers is 17, then the sum of the three terms surrounding the left and right terms must be 16 or less. Consider three consecutive numbers a, b, c such that $a + b + c = 17$. Let the number to the left of a be k. Assume that $a + b + k = 17$ also, then clearly $c = k$ which is impossible. Hence the result is proved. So we know that the maximum sum of the numbers used is $\frac{5 \times 17 \times 5 \times 16}{3} = \frac{5 \times 33}{3} = 55$. This is exactly the sum of the numbers of 1 to 10, hence the circle can be completed only if we have alternating 16s and 17s. Consider a, b, c, d, e, f, g as seven consecutive numbers in the circle. Then assuming $c + d + e = 17$ we have:

$$b + c + d = 16, \quad d + e + f = 16, \quad \text{so} \quad b + c = e + f \tag{1}$$

Now $a + b + c = 17, \quad e + f + g = 17$. However from equation (1) $a = g$. Hence it is impossible to arrange the numbers with maximum 17.



Also solved by Lisa Gotley, All Saints' Anglican School, Merrimac, Qld.

Q.910 A list of numbers $u_1, u_2, u_3, \ldots$ is defined as follows. The first number is $u_1 = 2$, and for $n \geq 1$ we let

$$u_{n+1} = 2u_n + \sqrt{3u_n^2 - 12}\,.$$

For example $u_2 = 2u_1 + \sqrt{3u_1^2 - 12} = 4$ and $u_3 = 2u_2 + \sqrt{3u_2^2 - 12} = 14$. Prove that all the numbers in the list are integers.

ANS. We have

$$(u_{n+1} - 2u_n)^2 = 3u_n^2 - 12.$$

Expanding and rearranging,

$$u_{n+1}^2 - 4u_{n+1}u_n + u_n^2 = -12.$$

Similarly,

$$u_n^2 - 4u_nu_{n-1} + u_{n-1}^2 = -12,$$

so

$$u_{n+1}^2 - 4u_{n+1}u_n + u_n^2 = u_n^2 - 4u_nu_{n-1} + u_{n-1}^2.$$

Adding $3u_n^2$ makes each side a perfect square:

$$(u_{n+1} - 2u_n)^2 = (2u_n - u_{n-1})^2.$$

Taking the positive square root of both sides,

$$u_{n+1} - 2u_n = 2u_n - u_{n-1},$$

since it is clear from the definition that $u_{n+1} \geq 2u_n$ for all n. Calculating u_2 from the given information we have

$$u_1 = 2,\ u_2 = 4,\ u_{n+1} = 4u_n - u_{n-1} \text{ for } n \geq 1.$$

Hence every u_n is an integer.

Solved by: Lisa Gotley, All Saints' Anglican School, Merrimac, Qld. Lisa also found the formula

$$u_n = (2+\sqrt{3})^{n-1} + (2-\sqrt{3})^{n-1}.$$

Q.911 (i) Two players alternate tossing a coin. The winner is the first player to have thrown both a head and a tail at least once each. What is the probability that the first player wins?

(ii) Two Martians alternate tossing a coin. The winner is the first player to have thrown heads, tails and paws at least once each. (Martian coins have three sides, all of which are equally likely to turn up.) What is the probability that the first player wins?

ANS. (i) Clearly neither player can win on his first toss. For $n \geq 2$, the first player wins on the nth toss (and not before) if and only if three conditions hold: all his previous $n-1$ throws turned up the same; all his opponent's previous throws turned up the same; and the first player's nth throw is different from all the previous ones. The probability of this happening is

$$\frac{2}{2^{n-1}} \times \frac{2}{2^{n-1}} \times \frac{1}{2} = \frac{1}{2^{2n-3}}$$

and so the first player's total chance of winning is

$$\begin{aligned}\sum_{n=2}^{\infty} \frac{1}{2^{2n-3}} &= \frac{1}{2} + \frac{1}{2^3} + \frac{1}{2^5} + \cdots \\ &= \frac{\frac{1}{2}}{1-\frac{1}{4}} \\ &= \frac{2}{3}.\end{aligned}$$

(ii) The probability that n throws of a Martian coin all turn up heads is $\left(\frac{1}{3}\right)^n$; and this is also the probability of "all tails" and of "all paws". So the total probability that only one result has been obtained after n tosses is $3 \times \left(\frac{1}{3}\right)^n$. Similarly, the probability that exactly two different results have been obtained after n tosses can be calculated as $3 \times \left(\frac{2}{3}\right)^n - 6 \times \left(\frac{1}{3}\right)^n$; the second term being subtracted because the first includes the possibilities "all heads", "all tails" and "all pairs", twice each. As in (i), the first player wins on his nth throw, and not earlier, if he has thrown exactly two different results in $n-1$ throws, his opponent has one or two different results in $n-1$ throws, and the first player throws the remaining result on his nth throw. The probability of this occurring is

$$\begin{aligned}&[3 \times (\tfrac{2}{3})^{n-1} - 6 \times (\tfrac{1}{3})^{n-1}][3 \times (\tfrac{2}{3})^{n-1} - 6 \times (\tfrac{1}{3})^{n-1} + 3 \times (\tfrac{1}{3})^{n-1}] \times \tfrac{1}{3} \\ &= [\tfrac{9}{2} \times (\tfrac{2}{3})^n - 18 \times (\tfrac{1}{3})^n][\tfrac{9}{2} \times (\tfrac{2}{3})^n - 9 \times (\tfrac{1}{3})^n] \times \tfrac{1}{3} \\ &= \tfrac{27}{4} \times (\tfrac{4}{9})^n - \tfrac{81}{2} \times (\tfrac{2}{9})^n + 54 \times (\tfrac{1}{9})^n.\end{aligned}$$

Note that this is 0, as it must be, when $n = 1$ or 2. Adding up three geometric series gives

$$\frac{\frac{27}{4} \times \frac{4}{9}}{1 - \frac{4}{9}} - \frac{\frac{81}{2} \times \frac{2}{9}}{1 - \frac{2}{9}} + \frac{54 \times \frac{1}{9}}{1 - \frac{1}{9}} = \frac{27}{5} - \frac{81}{7} + \frac{27}{4} = \frac{81}{140}$$

as the first player's total probability of winning.

Comment. Consider a similar game using a (six-sided) die. We can approach the problem by similar methods, but, as you might imagine, the algebra and arithmetic become horrendous. With the aid of the computer algebra system MAPLE I find the first player's probability of winning to be

$$\frac{6688879}{12660648}$$

which is a little under 0.53. So chances are fairly even in this case.

Solution of (i) from Lisa Gotley, All Saints' Anglican School, Merrimac, Qld.

***Q.912** I tell Jack and Jill each a positive integer. I tell both of them that the sum of the two positive integers is either 4 or 5, and ask each in turn whether they know the other's number.

Jack: "I don't know."
Jill: "I don't know either."
Jack: "I still don't know."
Jill: "Now I know!"

How *did* she know?

ANS. Solution by Lisa Gotley [with extra comments in square brackets].

If either Jack or Jill has a 4 then on their *first* go they will know that the other has a 1. So Jack holds a 1,2 or 3, as does Jill. [Note that not only we, but also Jack and Jill themselves, can deduce this fact.]

Jack says, on his first go, "I don't know", so if Jill holds the 1 she will know immediately that Jack has the 3. [She already knows he doesn't have the 4.] If Jack held a 1, Jill holds 3 or 4. When she says "I don't know" on her first turn, the possibility of 4 is eliminated. Jack would know she holds the 3. So Jack holds 2 or 3, as does Jill.

If Jack holds a 3, Jill holds a 1 or 2. However we already know [and so does Jack] that Jill can't hold 1, so Jack would know on his second turn that Jill holds a 3. Therefore Jack must hold a 2 and Jill a 2 or 3.

This is how Jill knew Jack had a 2.

*Lindsay Winkler, Newington College solved **Q.906, Q.912** and **Q.909** (i), (ii).

Continued from p.21

Q.922 Show how to cut a square of area 12 into seven pieces which can be rearranged to form either a 3×4 or a 2×6 rectangle.