Parabola Volume 30, Issue 2 (1994)

THE 3n + 1 PROBLEM

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Mathematics is full of unsolved problems and conjectures which attract the minds of many people, both amateurs and professionals. Unlike many mathematical problems, the $3n+1$ problem, (also known as the 'Collatz problem', or the 'Syracuse problem' or 'Ulam's problem' or 'Kakutani's problem') can be explained to anyone who can multiply and divide!

Choose any positive integer n. If n is even then divide it by 2. If n is odd, then multiply by 3 and add 1. Repeat the process until you get 1.

For example, if $n = 13$, we get the sequence

$$13, 40, 20, 10, 5, 16, 8, 4, 2, 1.$$

The case $n = 19$ takes a bit longer:

$$19, 58, 29, 88, 44, 22, 11, 34, 17, 52, 26, 13, \ldots, 1.$$

The 'problem' lies in the words 'until you get 1'. Is there any guarantee that the sequence will eventually reach 1? The answer is not known. A prize of £1000 was offered for a solution in 1982.

To give some idea of what happens for various values of n, let N be the number of terms in the sequence (excluding n) ending at 1. Here is a table of some values of N.

n	N	n	N
51	24	59	32
52	11	60	19
53	11	61	19
54	112	62	107
55	112	63	107
56	19	64	6
57	32	65	27
58	19	66	27

What can one do with an unsolved problem? One thing is to look at variations of it, which might be more tractable. For example, you could change the rules and multiply an odd term by 5 say and add 1 or see what happens to numbers of a certain form. This is precisely what I attempted and discovered the following result while playing with my calculator.

Theorem. *If k is odd and $2^k - 1$ has N terms in its sequence, then $2^{k+1} - 1$ has $(N+1)$ terms in its sequence.*

For example, $2^7 - 1 = 127$ requires 46 steps to reach 1, while $2^8 - 1 = 255$ requires 47 steps. (You might like to check this).

Proof. Let $a_0 = 2^k - 1$, k odd, then

$$a_2 = \frac{3(2^k - 1) + 1}{2} = 3.2^{k-1} - 1 \text{ which is odd}$$
$$a_4 = \frac{3(3.2^{k-1} - 1) + 1}{2} = 3^2.2^{k-1} - 1 \text{ which again is odd.}$$

Continuing the process, we reach $a_{2k} = 3^k - 1$ which is even, but **not** divisible by 4, since $3^k - 1 \equiv (-1)^k - 1 \equiv -2 \bmod 4 \not\equiv 0 \bmod 4$ (remember k is odd).

Hence $a_{2k+1} = \dfrac{3^k - 1}{2}$ is odd and $a_{2k+2} = \dfrac{3^{k+1} - 1}{2}$.

Now let $b_0 = 2^{k+1} - 1$, k odd, then by the above argument $b_{2k+2} = 3^{k+1} - 1$ which is even so $b_{2k+3} = \dfrac{3^{k+1} - 1}{2} = a_{2k+2}$.

Hence both sequences eventually reach the number $\dfrac{3^{k+1} - 1}{2}$ but $2^{k+1} - 1$ takes one extra step, and the theorem is proved.

Notice that this does **not** prove that either number does eventually reach 1.

I invite you to play with the problem yourself and see what you can find. For example, can you find the number(s) less than 10,000 which have the longest sequence? Are there other types of numbers whose sequence length have some special relationship? Is the theorem true for $n_1 = 2^k s - 1$ and $n_2 = 2^{k+1}s - 1$ with s odd? What if $s \equiv 1 \bmod 4$?

Experimenting in this way with difficult problems can often lead to some interesting results.