Parabola Volume 30, Issue 2 (1994)

UNSW SCHOOL MATHEMATICS COMPETITION 1994

SOLUTIONS

JUNIOR DIVISION

1. A large number of brown, green and yellow frogs are wandering around on an island. Whenever two frogs of different colours meet each other, they change immediately into two frogs of the third colour. More than two frogs never meet simultaneously. If there are initially 1993 brown frogs, 1994 green frogs and 1995 yellow frogs on the island, is it possible that at some future time all the frogs will have the same colour?

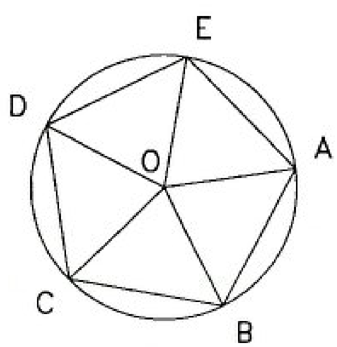
   Solution. To obtain frogs of just one colour we must first reach a situation in which two colours have equal numbers of frogs. When two frogs meet (and change colour), either the numbers in two colours both decrease by 1, and so their difference is unchanged; or one of the numbers decreases by 1 and the other increases by 2, so that the difference changes by 3. Since the differences are initially 1, 1 and 2 and they can only change by 3, the difference between two colours can never be 0. Thus it is impossible to get equal numbers of frogs of two colours, and hence impossible to get all frogs of one colour.

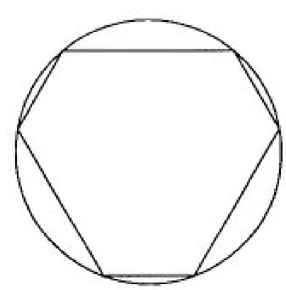
2. Are the following statements true or false? Prove your answers.

   (a) A pentagon inscribed in a circle and having all of its angles equal *must* have all of its sides equal.

   (b) A hexagon inscribed in a circle and having all of its angles equal *must* have all of its sides equal.

   Solution. (a) See the left-hand diagram below. Let $\angle OAB = \alpha$, $\angle OAE = \beta$. Then each of the angles of the pentagon is $\alpha + \beta$. Since $\triangle OAB$ is isosceles ($OA = OB$) we have $\angle OBA = \alpha$ and $\angle OBC = \beta$. Similarly $\angle OCB = \beta$ and $\angle OCD = \alpha$, and eventually we get $\angle OEA = \alpha$. But since $\triangle OAE$ is isosceles this means that $\alpha = \beta$. Therefore all five triangles in the figure are congruent, and the five sides of the pentagon are all equal.





   (b) Such a hexagon need not have all of its sides equal: see, for example, the right-hand diagram above.

3. Add up the natural numbers in the sets

$$\{1\}, \quad \{4,5,6\}, \quad \{11,12,13,14,15\}, \ldots,$$

where one natural number is added, two omitted, three added, four omitted and so on, and where n sets of numbers are taken altogether. Prove that your answer is correct.

Solution. The kth set contains $2k-1$ numbers, the last of which is

$$1+2+3+\cdots+(2k-1).$$

Summing an arithmetic progression, this last number is

$$\tfrac{1}{2}(2k-1)(1+2k-1) = k(2k-1) = 2k^2-k.$$

Hence the numbers in the kth set also form an arithmetic progression, and their sum is

$$\begin{aligned}
[(2k^2-k)-(2k-2)] + [(2k^2-k)-(2k-3)] + \cdots + [2k^2-k] \\
&= \tfrac{1}{2}(2k-1)[(2k^2-k)-(2k-2)+(2k^2-k)] \\
&= (2k-1)(2k^2-2k+1) \\
&= 4k^3-6k^2+4k-1 \\
&= k^4-(k-1)^4.
\end{aligned}$$

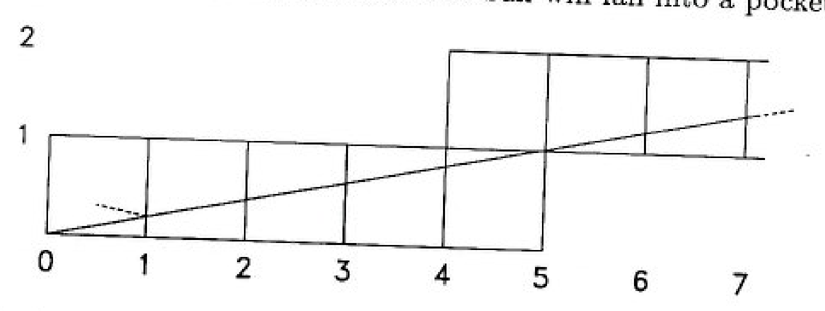
Therefore the sum of all n sets of numbers is

$$n^4-(n-1)^4+(n-1)^4-(n-2)^4+\cdots+2^4-1^4+1^4-0^4$$

which equals n^4 since everything else cancels.

4. A square billiard table with side length 1 metre has a pocket at each corner. A ball is struck from one corner and hits the opposite wall at a distance of $\frac{19}{94}$ metres from the adjacent corner. If the ball keeps travelling, how many walls will it hit before it falls into a pocket?

Solution. Imagine that instead of the ball being reflected off the side of the table, the ball keeps going and the table is reflected! The ball will fall into a pocket when it



reaches for the first time a point (m,n) where both m and n are integers. Now the points in the diagram at which the ball crosses a vertical line are

$$(1, \tfrac{19}{94}), \; (2, 2\times\tfrac{19}{94}), \; (3, 3\times\tfrac{19}{94}), \ldots$$

and since 19 and 94 have no common factor, the first such point at which the ball is also on a horizontal line is (94, 19). In reaching this point the ball will have crossed 93 vertical lines and 18 horizontal lines; thus, in the original problem, the ball will hit 111 walls before falling into a pocket.

5. An infinite list of positive numbers $a_0, a_1, a_2, a_3, \ldots$ has the property that

$$a_{n+1} = a_n + \frac{1}{a_n}$$

for $n = 0, 1, 2, \ldots$. Prove that a_{1994} is greater than 63.

Solution. Squaring both sides,

$$(a_{n+1})^2 = (a_n)^2 + 2 + \frac{1}{(a_n)^2} > (a_n)^2 + 2 .$$

Hence $(a_1)^2 > 2$, $(a_2)^2 > 4$, ..., $(a_{1994})^2 > 3988$ and we have

$$a_{1994} > \sqrt{3988} > 63 .$$

6. It is desired to write a given integer as the sum of four of its divisors (with repetition of divisors forbidden). For example, 24 can be expressed in two such ways:

$$24 = 12 + 8 + 3 + 1 = 12 + 6 + 4 + 2 .$$

Show that for any given integer there are no more than six ways of doing this, and find the smallest positive integer which has six solutions.

Solution. Suppose that $n = d_1 + d_2 + d_3 + d_4$, where

$$d_1 e_1 = d_2 e_2 = d_3 e_3 = d_4 e_4 = n .$$

Dividing each side of the first equation by n gives

$$1 = \frac{1}{e_1} + \frac{1}{e_2} + \frac{1}{e_3} + \frac{1}{e_4} ,$$

and so the problem is equivalent to that of writing 1 as a sum of four different *unit fractions* (that is, fractions with numerator 1).

First try

$$1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{p} + \frac{1}{q}$$

with $3 < p < q$. Collecting the numerical terms and clearing the denominators gives

$$pq = 6p + 6q .$$

If we now take everything to the left hand side and add 36 to both sides we obtain

$$pq - 6p - 6q + 36 = 36 ,$$

and the left hand side can be factorised to give

$$(p-6)(q-6) = 36 .$$

Thus $p-6$ is a factor of 36. However since $p-6 < q-6$ we must have $p-6 < 6$, and hence $p-6 = 1, 2, 3$ or 4. So we have four solutions $p = 7, 8, 9$ or 10, with $q = 42, 24, 18$ or 15 respectively.

Next try

$$1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{p} + \frac{1}{q}$$

with $4 < p < q$. The same procedure gives $(p-4)(q-4) = 16$ and leads to two more solutions $p = 5, 6$ with $q = 20, 12$.

There are no more solutions. To see this, note that apart from the cases we have already examined, the largest sum of four unit fractions including $\frac{1}{2}$ is $\frac{1}{2} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} = \frac{106}{105} \neq 1$, and the second largest is $\frac{1}{2} + \frac{1}{5} + \frac{1}{6} + \frac{1}{8} = \frac{119}{120}$, which is already too small; while if $\frac{1}{2}$ is not used then the largest possibility is $\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} = \frac{19}{20}$, which again is too small. Thus we have only six solutions

$$\begin{aligned} 1 &= \frac{1}{2} + \frac{1}{3} + \frac{1}{7} + \frac{1}{42} = \frac{1}{2} + \frac{1}{3} + \frac{1}{8} + \frac{1}{24} \\ &= \frac{1}{2} + \frac{1}{3} + \frac{1}{9} + \frac{1}{18} = \frac{1}{2} + \frac{1}{3} + \frac{1}{10} + \frac{1}{15} \\ &= \frac{1}{2} + \frac{1}{4} + \frac{1}{5} + \frac{1}{20} = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{12} \end{aligned}$$

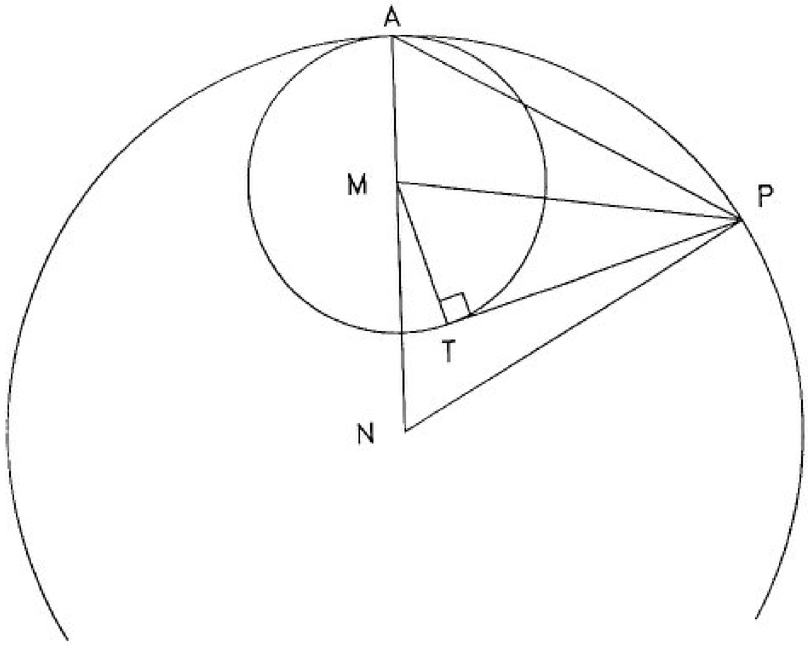
to the "four unit fractions" problem. Each of these will provide a solution of the original problem for any n which is a common denominator for the four fractions; therefore the original problem, for any n, will have six solutions or fewer. There will in fact be six solutions if and only if n is a common multiple of all twenty-four denominators $2, 3, 7, 42, \ldots, 6, 12$, that is, if and only if n is a multiple of 2520. Therefore 2520 is the smallest number for which six solutions are possible, and these solutions are

$$\begin{aligned} 2520 &= 1260 + 840 + 360 + 60 = 1260 + 840 + 315 + 105 \\ &= 1260 + 840 + 280 + 140 = 1260 + 840 + 252 + 168 \\ &= 1260 + 630 + 504 + 126 = 1260 + 630 + 420 + 210 . \end{aligned}$$

SENIOR DIVISION

1. A small circle is located inside a larger circle, with the two circles touching at the point A. P is a point on the large circle, and T is a point on the small circle such that PT is tangent to the small circle. Prove that provided $P \neq A$, the ratio of the lengths of PT and PA is the same for any point P on the large circle.

Solution. Let M, N be the centres of the circles, as shown; let r, R be the



radii of the smaller and larger circles respectively, and write $\angle MNP = \theta$. By the cosine rule in $\triangle ANP$ we have

$$PA^2 = 2R^2 - 2R^2\cos\theta = 2R^2(1-\cos\theta) \ .$$

Similarly, the cosine rule in $\triangle MNP$ gives

$$PM^2 = R^2 + (R-r)^2 - 2R(R-r)\cos\theta \ ,$$

and since $\angle MTP$ is a right angle, a little algebra leads to

$$PT^2 = PM^2 - r^2 = 2R(R-r)(1-\cos\theta) \ .$$

Hence

$$\frac{PT}{PA} = \sqrt{\frac{R-r}{R}}$$

which does not depend on θ, and is therefore the same for any position of P.

2. An infinite list of positive numbers $a_0, a_1, a_2, a_3, \ldots$ has the property that

$$a_{n+1} = a_n + \frac{1}{a_n}$$

for $n = 0, 1, 2, \ldots$. Prove that a_{1994} is greater than 63.

Solution. See problem 5 in the Junior Division.

3. Each of the numbers x_1, x_2, x_3, x_4, x_5 satisfies the inequality $-1 \le x_k \le 1$. What is the smallest possible value of

$$x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4 \ ?$$

What is the smallest possible value of

$$x_1x_2 + x_1x_3 + x_1x_4 + x_1x_5 + x_2x_3 + x_2x_4 + x_2x_5 + x_3x_4 + x_3x_5 + x_4x_5 \ ?$$

Solution. For the first part of the question, we have

$$x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4 = \tfrac{1}{2}\left[(x_1 + x_2 + x_3 + x_4)^2 - (x_1^2 + x_2^2 + x_3^2 + x_4^2)\right] .$$

Now certainly

$$(x_1 + x_2 + x_3 + x_4)^2 \geq 0 ,$$

and since each variable is between -1 and 1 we have also

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 \leq 4 .$$

Therefore

$$x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4 \geq \tfrac{1}{2}[0 - 4] = -2 .$$

Moreover, it is possible for the expression to reach this minimum value, for example, when $x_1 = 1$, $x_2 = -1$, $x_3 = 1$ and $x_4 = -1$.

For the second part, write

$$x_1x_2 + x_1x_3 + x_1x_4 + x_1x_5 + x_2x_3 + x_2x_4 + x_2x_5 + x_3x_4 + x_3x_5 + x_4x_5$$
$$= x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4 + (x_1 + x_2 + x_3 + x_4)x_5 .$$

If $x_1 + x_2 + x_3 + x_4 \geq 0$ then this expression takes its minimum value when x_5 takes its minimum value, that is, -1; while if $x_1 + x_2 + x_3 + x_4 \leq 0$ the expression takes its minimum value when x_5 takes its maximum value, 1. Thus we can find the minimum value of the expression by letting $x_5 = \pm 1$; and the same is true of the other four variables.

If all five variables equal 1 then clearly $x_1x_2 + \cdots + x_4x_5 = 10$. If four are 1 and one is -1, then four of the products are -1, six are 1 and the sum is 2; if three are 1 and two -1 then six products are -1 and four 1 for a total of -2; and the remaining cases give sums of -2, 2 and 10 again. So the minimum possible value is again -2, occurring (for example) when $x_1 = x_2 = x_3 = 1$ and $x_4 = x_5 = -1$.

4. Find a sixth-degree polynomial with integer coefficients which is a factor of $x^{15} + 1$.

Solution. For any odd integer n,

$$X^n + 1 = (X + 1)(X^{n-1} - X^{n-2} + \cdots - X + 1)$$

and so $X + 1$ is a factor of $X^n + 1$. In particular, let $X = x^5$ and $n = 3$: then $x^5 + 1$ is a factor of $x^{15} + 1$. In the same way, taking $X = x^3$ and $n = 5$ shows that $x^3 + 1$ is a factor of $x^{15} + 1$. Furthermore,

$$x^5 + 1 = (x + 1)(x^4 - x^3 + x^2 - x + 1) \quad \text{and} \quad x^3 + 1 = (x + 1)(x^2 - x + 1) ,$$

so $x^4 - x^3 + x^2 - x + 1$ and $x^2 - x + 1$ are factors of $x^{15} + 1$. Now suppose that these two polynomials themselves have a common factor. Then there is a (complex) number x such that

$$x^4 - x^3 + x^2 - x + 1 = 0 \quad \text{and} \quad x^2 - x + 1 = 0 \ ;$$

hence

$$x^4 - x^3 + x^2 = 0 \ \Rightarrow \ -x + 1 = 0 \ \Rightarrow \ x = 1 \ ,$$

which is impossible as $x^2 - x + 1$ is not zero when $x = 1$. Hence the quartic and the quadratic found above are factors of $x^{15} + 1$, and they themselves have no common factor, so their product

$$x^6 - 2x^5 + 3x^4 - 3x^3 + 3x^2 - 2x + 1$$

is a factor of $x^{15} + 1$.

5. Prove that, of the positive integers less than one million, there are at least 293 which cannot be written as a positive integer plus the sum of the squares of the digits of that integer.

Solution. For any positive integer n define n^* to be the sum of n and the squares of the digits of n. Essentially, we want to know how many values of n^* are less than one million. Clearly, if $n \geq 1\,000\,000$ then $n^* \geq 1\,000\,000$. Also, if $999\,707 \leq n < 1\,000\,000$ then the first three digits of n are nines and the fourth is seven or more, so

$$n^* \geq 999\,707 + 9^2 + 9^2 + 9^2 + 7^2 = 1\,000\,000 \ .$$

This leaves 999 706 values of n; these give 999 706 values of n^*, and, indeed, even some of these values may be greater than one million (for example, $999\,699^* = 1\,000\,141$) or may repeat each other (for example, $5^* = 22^*$). So there are at most 999 706 values of n^* less than one million, and therefore at least $999\,999 - 999\,706$, that is, 293 integers less than one million which are not equal to any value of n^*.

6. In Martian rules football, the score is made up of goals and behinds, each being worth a certain whole number of points. There are 35 scores which it is impossible for a team to get in total; one of these scores is 58. If a goal is worth more than a behind, how many points are scored for each?

Solution. Let g and b be the number of points scored for a goal and a behind. First observe that g and b have no common factor (otherwise there are infinitely many impossible totals), and $b > 1$ (otherwise there are no impossible totals). Now consider a score $s \geq gb$. Since g, b have no common factor we can find integers x, y such that

$$gx + by = s \ .$$

Thus scoring x goals and y behinds will amass a total of s points. Of course we need x and y to be non-negative – even in Martian rules you can't score negative goals! To achieve this, note that if $x < 0$ we can increase x by b and decrease y by g, giving a score

$$g(x + b) + b(y - g) = gx + by = s \ .$$

We can keep on increasing the value of x until it is non-negative. Similarly, if x is initially greater than b we can reduce it in steps of b until we get

$$gx + by = s\ , \quad 0 \le x < b\ ;$$

in this case we also have

$$by = s - gx > gb - gb = 0\ ,$$

so y, the number of behinds scored, is positive. This shows that it is possible to score any total not less than gb.

The possible scores less than gb are the following:

$$\begin{aligned} qb\ , &\quad \text{where } 0 \le q < g; \\ g + qb\ , &\quad \text{where } 0 \le q < (b-1)g/b; \\ &\vdots \\ (b-1)g + qb\ , &\quad \text{where } 0 \le q < g/b. \end{aligned}$$

There are g possibilities in the first group, $g + 1$ in the second and the last combined, $g + 1$ in the third and the second last combined, and so on, for a total of

$$g + \tfrac{1}{2}(b-1)(g+1)\ ;$$

all these possibilities are different since g and b have no common factor. Hence the total number of impossible scores is

$$gb - \left(g + \tfrac{1}{2}(b-1)(g+1)\right) = \tfrac{1}{2}(g-1)(b-1)$$

and it is given that this equals 35. So $(g-1)(b-1) = 70$ and we have the possibilities

$$\begin{aligned} g - 1 &= 70\ , & b - 1 &= 1\ ; \\ g - 1 &= 35\ , & b - 1 &= 2\ ; \\ g - 1 &= 14\ , & b - 1 &= 5\ ; \\ g - 1 &= 10\ , & b - 1 &= 7\ . \end{aligned}$$

The second of these gives $g = 36$, $b = 3$, which is impossible as we know that g and b have no common factor; and the third is ruled out similarly. Finally, the first would give $b = 2$, and this must be eliminated since it is given that a total of 58 cannot be scored. Therefore a goal is worth 11 points and a behind is worth 8.