SOLUTIONS TO PROBLEMS 913-922

Q.913 S is a collection of numbers. At least half of the numbers in S are even, at least two thirds are multiples of 3, and at least six sevenths are multiples of 7. Prove that at least one of the numbers in S is a multiple of 42.

ANS. The sum of the three proportions mentioned in the question is

$$\frac{1}{2}+\frac{2}{3}+\frac{6}{7}=\frac{85}{42}>2.$$

Therefore there must be a number in the set having all three properties (otherwise the proportions could add up to at most 2). That is, there is a number divisible by $2,3$ and 7, and hence by 42.

Q.914 In the following addition, the letters $A,B,\ldots,I$ represent single digits, all different.

$$\begin{array}{rr} & A\,B\,C \\ + & D\,E\,F \\ + & G\,H\,I \\ \hline & 1\,9\,9\,4 \end{array}$$

In how many ways can the sum be reconstructed?

ANS. First we determine which digit is not used. The *digital sum* of a number is obtained by adding the digits of the number, adding the digits of the result, and so on until a single digit remains. Now the digital sum of $A,B,\ldots,I$ equals the digital sum of 1994, which is 5. But the digital sum of all ten digits is 9, and so the unused digit is 4.

Allowing for the possibility that digits may be carried in the addition, the three columns of the sum must add up to

$$19,9,4 \quad \text{or} \quad 19,8,14 \quad \text{or} \quad 18,19,4 \quad \text{or} \quad 18,18,14.$$

Since the sum of all nine digits is 41 we may immediately rule out the first and last of these.

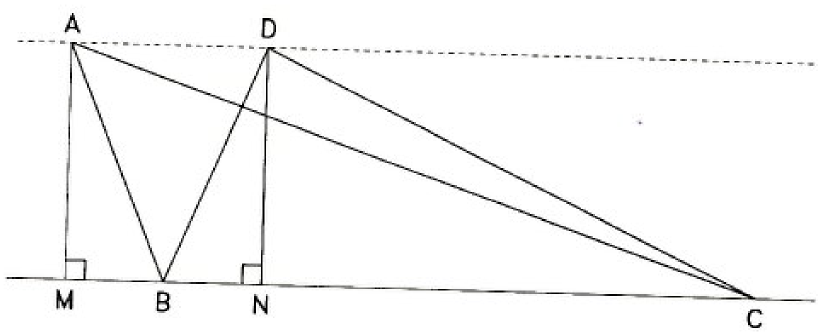
In the third case, three different digits adding up to 4 can only be 3,1,0; then the digits adding up to 19 are 9,8,2 or 8,6,5; and the digits adding up to 18 are the remaining three. So here there are two possibilities. In the second case we have five possibilities:

digits adding to 8	digits adding to 14
7,1,0	9,3,2
	6,5,3
6,2,0	8,5,1
5,3,0	7,6,1
5,2,1	8,6,0

Thus we can choose appropriate columns of digits in seven ways. But each column may be arranged in six different orders, which gives $7 \times 6 \times 6 \times 6 = 1512$ reconstructions of the sum.

Q.915 A triangle has sides of length 13, 30 and 37. Find all possible triangles with sides 13 and 30, a different third side, and the same area as the given triangle.

ANS. Draw $\triangle ABC$ as shown below, with $AB = 13$, $BC = 30$, $AC = 37$. Another triangle $\triangle DBC$ on the same base will have the same area as $\triangle ABC$ if and only if $AD \parallel BC$. Therefore the only possibility is that shown, with $BD = 13$.



Now let $MB = x$. By Pythagoras' theorem,

$$AM^2 = AB^2 - x^2 = AC^2 - (BC + x)^2,$$

and so

$$13^2 - x^2 = 37^2 - (30 + x)^2.$$

Expanding this equation, the x^2 terms drop out and we easily find $x = 5$. Clearly $BN = BM$, and $DN = AM$, so

$$DC^2 = DN^2 + NC^2 = 12^2 + 25^2 = 769.$$

Thus the required triangle $\triangle DBC$ has sides 13, 30 and $\sqrt{769}$.

Q.916 Find two ten-digit numbers such that the first digit of each number is the number of ones in the other, the second digit in each is the number of twos in the other, and so on, the tenth digit in each being the number of zeros in the other.

ANS. A neat method (though not guaranteed to work) of approaching this problem is as follows. Choose any ten-digit number and write down the number of ones, twos, ..., nines, zeros among its digits. This gives a new ten-digit number; repeat the procedure. If we ever reach – and we often will – a cycle of two numbers, then we have found a solution to the problem. Try, for example,

$$\begin{aligned} 1234554321 &\rightarrow 2222200000 \\ &\rightarrow 0500000005 \\ &\rightarrow 0000200008 \\ &\rightarrow 0100000108 \\ &\rightarrow 2000000107 \\ &\rightarrow 1100001007 \\ &\rightarrow 3000001006 \\ &\rightarrow 1010010007 \\ &\rightarrow 3000001006 \end{aligned}$$

Hence 1010010007 and 3000001006 form a suitable pair. (Not all starting numbers will be successful. This is something you might like to investigate.)

Q.917 We play a game as follows: a coin is tossed repeatedly; for each head you score one point and for each tail I score one point. The first to seven points wins \$10 from the other. When the score reaches 6-4 in your favour, I note that you have won 6 points out of 10, and offer you \$6 to abandon the game. Should you accept? What if the winner were to be the first player to reach ten points, all other conditions remaining the same?

ANS. In the first case you should not accept. The only way you can lose is for three successive tails to be tossed; the chance of this happening is $\frac{1}{8}$. Therefore the "average" amount you could expect to win if the game continues is

$$\$\left(\frac{7}{8}\times 10+\frac{1}{8}\times(-10)\right)=\$7.50,$$

and clearly you should not give this away in return for \$6.

In the second case, you lose if and only if three or fewer of the next 9 tosses are heads. Using the binomial distribution, this occurs with probability

$$\left[\binom{9}{0}+\binom{9}{1}+\binom{9}{2}+\binom{9}{3}\right]\times\left(\frac{1}{2}\right)^9=\frac{65}{256}.$$

Your expectation from the game is thus

$$\$\left(\frac{191}{256}\times 10+\frac{65}{256}\times(-10)\right)=\$4.92$$

and you are better off accepting the \$6.

Q.918 The digits $1,2,3,4$ are to be arranged to form a number, using *no* extra symbols. Two methods are permitted: joining digits to form a multi-digit number, and using power notation. For example some possible constructions are

$$123^4,\ 41^{32}\quad\text{and}\quad 3^{21^4}.$$

(Note that the last means $3^{(21^4)}$, not $(3^{21})^4$: such an expression is evaluated from the top down.) What is the largest number that can be formed under these conditions?

ANS. The largest possible number is $2^{3^{41}}$. To verify this we consider the various patterns in which the four digits can be arranged. Here a string of digits denotes a single number, not a product. We have

$$
\begin{aligned}
abcd &\le 4321 < 2^{13} < 2^{3^3} < 2^{3^{41}} \\
abc^d &< 432^4 < (2^9)^4 = 2^{36} < 2^{3^4} < 2^{3^{41}} \\
ab^{cd} &< 43^{43} < (2^6)^{43} = 2^{258} < 2^{3^6} < 2^{3^{41}} \\
ab^{c^d} &< 43^{4^4} < (2^6)^{4^4} = 2^{6\times 4^4} < 2^{3^2\times 9^4} = 2^{3^{10}} < 2^{3^{41}} \\
a^{bcd} &< 4^{432} = 2^{864} < 2^{3^7} < 2^{3^{41}} \\
a^{bc^d} &< 4^{43^4} = 2^{2\times 43^4} < 2^{3\times(3^4)^4} = 2^{3^{17}} < 2^{3^{41}} \\
4^{b^{cd}} &< 4^{3^{32}} = 2^{2\times 3^{32}} < 2^{3^{33}} < 2^{3^{41}}
\end{aligned}
$$

Now for the hard parts! If $b = 1$ then $3^{b^{cd}} = 3 < 2^{3^{41}}$, so we need only consider

$$3^{4^{21}} < 2^{2\times 4^{21}} = 2^{2^{43}} < 2^{2^{45}} = 2^{8^{15}} < 2^{9^{15}} = 2^{3^{30}} < 2^{3^{41}}$$

and

$$3^{2^{41}} < 2^{2\times 2^{41}} = 2^{2^{42}} < 2^{2^{43}} < 2^{3^{41}}.$$

Since one of the available digits is 1, any "stack" of four single digits will reduce to a, a^b or a^{b^c}; and we have

$$a \le a^b \le a^{b^c} \le 4^{4^4} = 2^{2\times 4^4} = 2^{2^9} < 2^{3^{41}}.$$

Finally we have to consider $2^{4^{31}}$. This can be brought within the range of a calculator by taking lorarithms twice:

$$2^{4^{31}} < 2^{3^{41}} \quad \text{iff} \quad 4^{31} < 3^{41} \quad \text{iff} \quad 31\log 4 < 41\log 3.$$

Calculation shows that 31 log 4 is indeed the smaller of these two numbers. Hence $2^{3^{41}}$ is the largest number obtainable.

Q.919 The Australian No-Hopers Party (ANHP) has 25 members in Parliament, including exactly two who live in Canberra, ride bicycles and own whiteboards. It is known that

(a) if the number of ANHP members who live in Canberra and own whiteboards is greater than 4 or the number who ride bicycles but do not live in Canberra is greater than 3, then the total number who own whiteboards is 15;

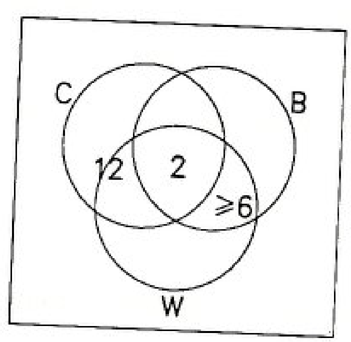
(b) if the number who ride bicycles is less than 9 or the number who do not own whiteboards is greater than 7, then the number who both ride bicycles and own whiteboards is 8;

(c) if the number who live in Canberra but do not ride bicycles is not equal to 12 or the number who ride bicycles and own whiteboards is less than 8, then the total number who ride bicycles is 4.

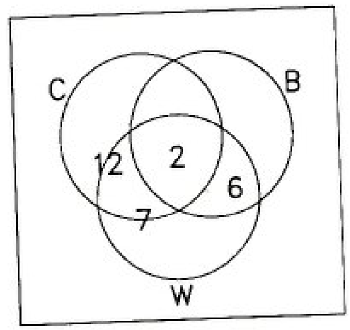
If you knew the total number of ANHP members who ride bicycles, you could calculate how many live in Canberra. Determine how many members of the ANHP own whiteboards, how many live in Canberra and how many ride bicycles.

ANS. *Notation.* Write c, b, w for the number of $ANHP$ members who live in Canberra, ride bicycles and own whiteboards respectively, and $\bar{c}, \bar{b}, \bar{w}$ for those who do not. Combinations of properties will be indicated by juxtaposition: for example, $c\bar{w}$ denotes the number of members who live in Canberra and do not own whiteboards.

Suppose $bw < 8$. Then, from (c), $b = 4$; so $b < 9$ and (b) tells us that $bw = 8$. This is a contradiction. Hence $bw \geq 8$. But $bw \leq b$ (the number who ride bicycles *and* own whiteboards cannot be more than the number who ride bicycles in the first place), so $b \neq 4$. Therefore, from property (c), $c\bar{b} = 12$. We can represent the information we have so far in a Venn diagram:



The 12 on a borderline indicates that we do not know exactly how these 12 members are distributed in the two regions. Clearly $\bar{c}b \geq 6 > 3$, so by (a) we have $w = 15$. Since there are altogether 25 members, $\bar{w} = 10 > 7$ and condition (b) gives $bw = 8$. This leads to an improved Venn diagram



Finally, it is stated that "if you knew the total number of $ANHP$ members who ride bicycles, you could calculate how many live in Canberra". This only makes sense if $b = 8$, so that we can put $cb\bar{w} = \bar{c}b\bar{w} = 0$ in the above diagram.

Thus

$$b = 8, \quad c = 12 + 0 + 2 = 14, \quad w = 7 + 2 + 6 = 15.$$

Q.920 The well-known Fibonacci sequence consists of the numbers $F_1, F_2, F_3, \ldots$ where $F_1 = F_2 = 1$ and each subsequent number in the list is the sum of the previous two: that is, $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. Show that one of the first million Fibonacci numbers is a multiple of 997.

ANS. Consider the sequence obtained from the Fibonacci sequence by taking the remainder of each number on division by 997. In this sequence there are only 997^2 possible pairs, and since $997^2 < 1000000$, the first million terms must contain a repeated pair. If, say, the pair F_n, F_{n+1} repeats F_m, F_{m+1} then

$$F_{n-1} = F_{n+1} - F_n = F_{m+1} - F_m = F_{m-1}$$

and so F_{n-1}, F_n is already a repetition of an earlier pair. Hence the repetitions go right back to the beginning of the sequence, and there exists n such that the