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DIVINE PERFECTION

Peter Brown

The ancient Greeks thought about numbers in quite a different way to us. Quite apart from the difficulties they had with irrational numbers, many Greek mathematicians and philosophers regarded the positive integers as having almost magical properties. Pythagoras (c.540–400BC), whose famous theorem is known by almost everybody, certainly believed this and was one of the first to begin to classify the positive integers into different types, the simplest classifications being even or odd, (there was some debate in antiquity as to whether 1 was odd and whether 2 was even!), prime or composite, square or oblong etc. Pythagoras left no written records of his work and the little we know about him comes mostly from the writings of the philosopher Plato (427–347BC), who wrote a vast amount of material (most of which survived), including quite a deal about mathematics. Plato's view of mathematics, indeed his whole view of the world, was that numbers and geometrical figures were not concrete objects which existed in the real world, but **ideas** or **forms** (as he called them), which did have real existence, but not in the visible world. The forms existed in a sort of 'cyberspace' called the world of forms, which could only be reached using our minds (or 'souls' as Plato would have said). For example, we define a point as having position but no dimension. In reality, when I 'draw' a point, it clearly is a two (three?) dimensional object as we can see by magnifying it. Plato would say that the point I draw is simply an imperfect representation of the 'real point' which lies in the world of forms, and does in fact have position but no dimensions in that world.

In the **Republic** he says

'Although mathematicians use visible figures and argue about them, they are not thinking of these figures (i.e. the ones they have drawn) but of those things (i.e. forms) which the figures represent; thus it is the square itself and the diagonal itself which are the matter of their arguments, not that which they draw.'

By the first century A.D., the mystical view of numbers became even more widespread, as did the classification of positive integers. This can be seen in the works of Theon of Smyrna and Nichomachos (100 AD), who devote several pages of their works to discussing the philosophical 'meaning' of the numbers 1 and 2.

Among the many different types of classifications, the Greeks divided numbers into three types: deficient, perfect and abundant. A number is called **deficient** if the sum of its proper divisors is less than the given number. (The set of **proper divisors** of a number consists of all the divisors except the number itself). For example 8 is deficient since $1+2+4=7<8$. Obviously all prime numbers are deficient. You might like to prove that the sum of the proper divisors of 2^n (n a positive integer) is 2^n-1 so 2^n is always deficient. An **abundant** number has its proper divisor sum greater than the number itself, for example 12;

$$1+2+3+4+6=16>12.$$

It was once believed that every abundant number must be even, but you can show this to be false using the number 945.

In between these two extremes are **perfect** numbers, that is, numbers which are exactly the sum of their proper divisors. The smallest examples are 6 and 28

$$6=1+2+3$$

$$28=1+2+4+7+14.$$

These numbers were held in great esteem by the ancient Greeks who regarded them as having a very elevated position in the world of forms.

Euclid (in **Book 9**) actually found (or at least recorded) a formula which gives even perfect numbers. It says: If $M=2^n-1$ is prime then

$$P=2^{n-1}(2^n-1) \text{ is a perfect number.}$$

Putting n to be 2 gives $M=3$ which is prime and $N=6$ while $n=3$ gives $M=7$ which is prime and $N=28$. If you put $n=4$ then $M=15$ which is not prime and so doesn't give a perfect number.

If you try $n=5,6,7,8,9,10$ you only get perfect numbers for $n=5$ and 7 so you might guess that 2^n-1 is prime when n is prime, but unfortunately this is false, since $2^{11}-1=2047=23\times 89$. Euclid believed that his formula gave **all** the even perfect numbers but couldn't prove it. In fact it was not until the 18th century that a proof was discovered by the famous Swiss Mathematician Leonard Euler (1707–1783).

Before going into more detail about perfect numbers, we need some mathematics. We use the symbol $\sigma(n)$ (sigma of n) to represent the sum of **all** the divisors of a number n. For example,

$$\sigma(18) = 1 + 2 + 3 + 6 + 9 + 18 = 39.$$

Clearly if p is a prime then $\sigma(p) = 1 + p$, and $\sigma(p^n) = 1 + p + p^2 + \ldots + p^n = \dfrac{p^{n+1} - 1}{p - 1}$ using geometric series. Can we find a formula (or procedure) to find $\sigma(n)$ for a general integer n?

Suppose $n = 16200 = 2^3.3^4.5^2$. Look at the effect of multiplying out

$$(1 + 2 + \underline{2^2} + 2^3)(1 + 3 + 3^2 + \underline{3^3} + 3^4)(1 + \underline{5} + 5^2).$$

Every factor of 16200 will appear in the expanded form **exactly once**, for example, $2^2.3^3.5$ is a factor and can be obtained from multiplying the numbers underline, and in no other way. Hence, the product of the three brackets will be equal to the sum of all the divisors of 16200, i.e. $\sigma(16200)$. So, to find $\sigma(n)$, factorise n into its prime factors $n = p_1^{\alpha_1} \ldots p_r^{\alpha_r}$ and use

$$\sigma(n) = (1 + p_1 + p_1^2 + \cdots + p_1^{\alpha_1})(1 + p_2 + p_2^2 + \cdots + p_2^{\alpha_2}) \cdots (1 + p_r + p_r^2 + \cdots + p_r^{\alpha_r}).$$

For example, $\sigma(24) = \sigma(2^3.3)$

$$= (1 + 2 + 2^2 + 2^3)(1 + 3) = 60.$$

Returning now to the problem of perfect numbers, suppose $2^n - 1$ is prime and $N = 2^{n-1}(2^n - 1)$. Then

$$\sigma(n) = (1 + 2 + \cdots + 2^{n-1})(1 + (2^n - 1))$$

$= (2^n - 1)2^n$ (using the formula for sum of a G.P.). To find the sum of the proper divisors of N, we need to subtract $2^{n-1}(2^n - 1)$ giving

$$(2^n - 1)2^n - 2^{n-1}(2^n - 1) = 2^{n-1}(2^n - 1)(2 - 1) = 2^{n-1}(2^n - 1) = N.$$

Hence N is a perfect number. To show that **all** even perfect numbers are given by this formula is more difficult. One obvious question that arises is: are there any odd perfect numbers? The answer is not known but it has been shown that if they exist, they must be **very** large, in fact, greater than 10^{200}.

The key to finding even perfect numbers is to find an n such that $2^n - 1$ is prime. Primes of this form are called Mersenne primes, named after Mersenne, an early 17th century mathematician. Quite a few such primes are known. For example in 1992, the number $2^{756839} - 1$ was shown to be prime and another one, $2^{859433} - 1$, was discovered in 1994. It is not known if there are infinitely many Mersenne primes (and thus infinitely many perfect numbers). In searching for Mersenne primes, the following fact is very important. If $2^n - 1$ is prime then n must be prime. To see this, suppose n is composite, then $n = a.b$ where neither a nor b is 1. Then

$$2^n - 1 = 2^{ab} - 1 = (2^a)^b - 1 = (2^a - 1)((2^a)^{b-1} + \cdots + 1)$$

and so $2^n - 1$ is not prime. (You can check this by using the sum of a G.P.) So, when searching for Mersenne primes, we only look at values of n which are prime. (Remember that the converse is false, e.g. $n = 11$ does not yield a Mersenne prime).

Here is a table showing some of the smaller Mersenne primes and the corresponding perfect number

n	m	P
5	31	496
7	127	8128
13	8191	33550336
17	13107	858986056
19	524287	137438691328

As you can see, they get big very quickly. By the way, do you notice that these perfect numbers always end in 6 or 8? This, in fact, is always true. You might like to try and prove this.

Among the many other interesting properties of perfect numbers is the following fact.

The sum of the reciprocals of all the divisors of any perfect number is 2, for example, using the perfect number 6

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 2.$$

For the perfect number 28

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{4}+\frac{1}{7}+\frac{1}{14}+\frac{1}{28}=2.$$

It is not too difficult to prove this in general. Can you find a number, such that the sum of reciprocals of all its divisors is 3? (672 works, can you find a smaller one?).

In addition to perfect numbers, the ancient Greeks, as far back as Pythagoras, looked for **amicable** (or friendly) numbers. An amicable pair of numbers has the property that each is the sum of the proper divisors of the other.

The smallest such pair is

$$220 = \sigma(284) - 284$$

$$284 = \sigma(220) - 220.$$

Numerologists have had a lot of fun with these numbers, for example in Genesis 32:14 Jacob gives 220 goats to Esau, and some bible scholars have seen this as symbolic of Jacob seeking the friendship of Esau. Many other pairs of amicable numbers are known, (more than a thousand in fact). Some examples are 1184 and 1210; 2620 and 2924.

No simple formula which generates all amicable pairs is known, but the Arabian mathematician Thabit ben Korrah give the following formula which generates some of these pairs.

If $p = 3.2^n - 1$, $q = 3.2^{n-1} - 1$ and $r = 9.2^{2n-1} - 1$ are **all** prime, then $2^n pq$ and $2^n r$ are amicable. You might like to try and prove this result (it is not hard).

For example, putting $n = 4$ gives

$$p = 47, \ q = 23, \ r = 1151$$

which are all prime, so $2^4 pq = 17296$ and $2^4 r = 18416$ are amicable.

It is not known whether or not there are infinitely many pairs of amicable numbers.

Another interesting set of numbers starts with 12496.

$$\sigma(12496) - 12496 = 14288$$

$$\sigma(14288) - 14288 = 15472$$

$$\sigma(15472) - 15472 = 14536$$

$$\sigma(14536) - 14536 = 14264$$

$$\text{and} \quad \sigma(14264) - 14264 = 12496$$

which is back to where we started. Such numbers are called **sociable** numbers, or amicable numbers of order 5.

There is, in fact, a lovely set of amicable numbers of order 28 which starts with 14316. You might like to find the other 27 numbers in the list.

Nowadays, we tend not to regard numbers as having magical properties (although some numerologists and fortune tellers do) and so although perfect and amicable numbers are interesting they do not hold the same place of importance in mathematics that they might have had in Plato's day. The search for large Mersenne primes is now thought of as more of a game than serious mathematics, although there is still research going on in problems related to perfect numbers and mathematicians can never be satisfied until the many outstanding unsolved problems relating to perfect and amicable numbers have been solved.

"PUZZLE"

Translate the following from 17th century Latin. No credit if you actually understand Latin!

"Cubum autem in duos cubos, aut quadratoquadratum in duos quadratoquadratos et generaliter nullam in infinitum ultra quadratum potestatem in duos eiusdem nominis fas est diuidere cuius rei demonstrationem mirabilem sane detexi. Hanc marginis exiguitas non caperet."

See solution on page 16.