## SOLUTIONS TO PROBLEMS 923–930

**Q.923** On an island there are 50 brown, 57 green, 62 yellow and 68 red frogs. Whenever three frogs of three different colours meet they change immediately into two frogs of the fourth colour. Later on it is observed that all frogs on the island have the same colour. Which colour, and what is the maximum possible number of frogs on the island at this stage?

**ANS.** First note that when three frogs meet they change into two frogs: therefore the number of frogs on this island is not constant, but decreases by one whenever three frogs of different colours meet.

At each meeting, the number of frogs of one colour increases by two while that of each other colour decreases by one. Hence the difference in numbers between any two colours can only change by 3 (if at all). So the three colours of frogs which eventually have no members must initially have differed by multiples of 3. These three can only be 50, 62 and 68; thus, the green frogs survived.

Now let $b$ be the number of meetings in which two brown frogs were created, and similarly define $g, y$ and $r$. Then the total number of red frogs lost in the course of all these meetings is $b+g+y-2r$, and this must equal 68 since no red frogs were left. Adding $3r$ to both sides,

$$b+g+y+r = 68+3r. \tag{1}$$

Since one frog is lost at each meeting, the final number of frogs is

$$50+57+62+68-(b+g+y+r) = 169-3r.$$

So this final number cannot be more than 169. To check that 169 is in fact possible we first consider the number of yellow and brown frogs lost. As above, we find

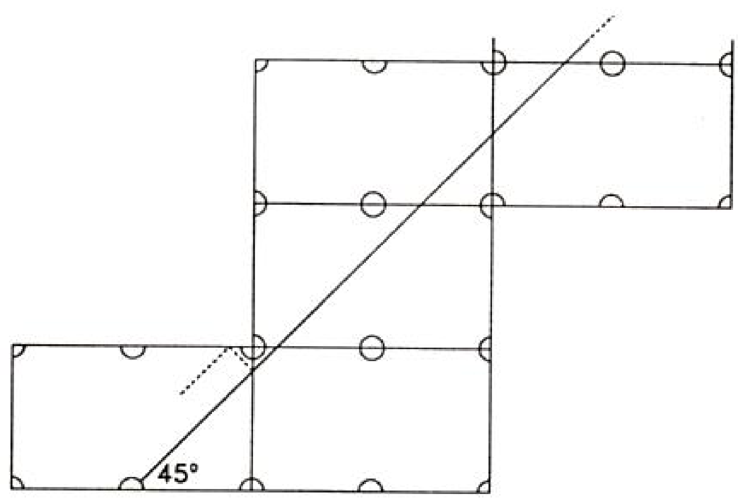
$$b+g+y+r = 62+3y \tag{2}$$

$$b+g+y+r = 50+3b. \tag{3}$$

To obtain the maximum number of frogs remaining we must take $r = 0$; then (1), (2), (3) yield $y = 2,\ b = 6,\ g = 60$. It is now easy to confirm that starting with $50, 57, 62, 68$ frogs we can create two pairs of yellow frogs to give $48, 55, 66, 66$; then six pairs of brown frogs to give $60, 49, 60, 60$; and finally, sixty pairs of green frogs to give $0, 169, 0, 0$.

**Q.924** A billiard table has dimensions $a \times b$, where $a$ and $b$ are integers. There is a pocket at each corner, and another halfway along each side of length $a$. A ball is hit from one of the centre pockets at an angle of $45°$ to the side of the table. If the ball keeps travelling, how many times will it hit a wall before falling into a pocket?

**ANS.** As in the competition problem, imagine that the table is reflected and the ball keeps going. It will fall into a pocket when it has




travelled vertically a distance of $mb$, for some integer $m$, and horizontally $n\left(\frac{1}{2}a\right)$, since pockets occur every $\frac{1}{2}a$ in the horizontal direction. Since the ball's trajectory is at $45°$ to the side of the table, $mb = \frac{1}{2}na$ and so

$$\frac{m}{n} = \frac{a}{2b}.$$

We need the smallest values of $m$ and $n$ satisfying this equation; hence we must reduce the right-hand side to lowest terms. Let $g$ be the greatest common divisor of $a$ and $2b$; then $m = \frac{a}{g}$, $n = \frac{2b}{g}$.

How many walls does the ball hit? Clearly in travelling a distance $mb$ vertically, $m-1$ horizontal walls are touched (the $m$th not counting as the ball falls into the pocket). In travelling $\frac{1}{2}a$, $a$, $\frac{3}{2}a$, $2a$, $\frac{5}{2}a, \ldots$ horizontally, the ball hits $0, 1, 1, 2, 2, \ldots$ vertical walls, and it is easy to see that in travelling $\frac{1}{2}na$, the number of walls hit is $\left[\frac{1}{2}n\right]$, where the square brackets indicate that any remaining half is rounded down. Using the values of $m$ and $n$ found above, the total number of walls struck is

$$\frac{a}{g} + \left[\frac{b}{g}\right] - 1,$$

where $g$ is the g.c.d. of $a$ and $2b$.

**Q.925** Show that in order to write an odd number as a sum of its divisors (without repetitions), at least 9 divisors must be used.

ANS. If $n$ is odd then all of its divisors are odd. Thus to write $n$ as a sum of divisors, an odd number of terms are required, as an even number of odd numbers add up to an even sum. As in problem 6 from the Junior Division of the UNSW/IBM competition, we need to write 1 as the sum of unit fractions

$$1 = \frac{1}{e_1} + \frac{1}{e_2} + \cdots + \frac{1}{e_k}$$

where $e_1, e_2, \ldots, e_k$ are odd, $e_1 < e_2, \ldots < e_k$ and $k$ is also odd. This will take rather a lot of computation, but here goes ...

First, five terms are not sufficient as they will give a maximum sum of

$$\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} = \frac{3043}{3465}$$

which is less than 1.

Next, consider the possibility of using seven terms. The fractions $\frac{1}{3}$, $\frac{1}{5}$, $\frac{1}{7}$, $\frac{1}{9}$ and $\frac{1}{11}$ must all be used, for if even the least of them is missing the maximum attainable is

$$\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\frac{1}{13}+\frac{1}{15}+\frac{1}{17}=\frac{68899}{69615}<1.$$

Also, either $\frac{1}{13}$ or $\frac{1}{15}$ must be used, otherwise the sum cannot exceed

$$\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\frac{1}{11}+\frac{1}{17}+\frac{1}{19}=\frac{1107629}{1119195}<1.$$

Hence we need either

$$\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\frac{1}{11}+\frac{1}{13}+\frac{1}{m}=1$$

or

$$\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\frac{1}{11}+\frac{1}{15}+\frac{1}{m}=1;$$

and a final burst of arithmetic gives $m = 22\frac{583}{2021}$ and $m = 18\frac{27}{191}$ respectively. These are both impossible as $m$ must be an integer. So the problem is impossible with seven factors, and we need at least nine. If you've not had enough arithmetic yet you could investigate this case; otherwise you could join me in accepting the authority of *The Penguin Dictionary of Curious and Interesting Numbers* by David Wells (Penguin, 1986), which states that there are four solutions with nine terms and gives the example

$$3465 = 1155 + 693 + 495 + 385 + 315 + 231 + 99 + 77 + 15.$$

**Q.926** If $n$ is the number whose digits are a two followed by 1994 threes, find the digits of $n^2$.

**ANS.** We have

$$\begin{aligned} n &= 2\overbrace{3\,3\cdots 3}^{1994\text{ digits}} \\ &= \frac{1}{3}\times 6\overbrace{9\,9\cdots 9}^{1994\text{ digits}} \\ &= \frac{1}{3}\times(7\times 10^{1994}-1) \end{aligned}$$

and so

$$
\begin{aligned}
n^2 &= \frac{1}{9}(49 \times 10^{3988} - 14 \times 10^{1994} + 1) \\
&= \frac{1}{9}\left((45+4) \times 10^{3988} + (-40+18+8) \times 10^{1994} + (-80+81)\right) \\
&= 5 \times 10^{3988} + \frac{4}{9} \times (10^{3988} - 10^{1995}) + 2 \times 10^{1994} \\
&\qquad + \frac{8}{9} \times (10^{1994} - 10) + 9.
\end{aligned}
$$

Noting that

$$
\begin{aligned}
10^{3988} - 10^{1995} &= (10^{1993} - 1) \times 10^{1995} \\
&= \overbrace{99\cdots9}^{1993}\,\overbrace{00\cdots0}^{1995},
\end{aligned}
$$

we have

$$
\begin{aligned}
n^2 = \quad & \overbrace{500\cdots0000\cdots00}^{3988} \\
+ \quad & \overbrace{44\cdots4}^{1993}\overbrace{000\cdots00}^{1995} \\
+ \quad & \overbrace{200\cdots00}^{1994} \\
+ \quad & \overbrace{88\cdots80}^{1993} \\
+ \quad & 9 \\
= \quad & 5\overbrace{44\cdots4}^{1993}2\overbrace{88\cdots8}^{1993}9\ .
\end{aligned}
$$

That is, $n^2$ begins with a five, followed by 1993 fours, a two, 1993 eights and a nine.

**Q.927** Five line segments are given, such that any three of them will form a triangle. Show that at least one of these triangles is acute-angled.

**ANS.** First recall two basic facts: (i) in a triangle any side is shorter than the sum of the other two sides; (ii) in a non-acute (that is, right-angled or obtuse) triangle with sides $x \geq y \geq z$, we have $x^2 \geq y^2 + z^2$. Now let the five given segments, in decreasing order of length, be $a, b, c, d$ and $e$. Then either the triangle $a, b, c$ or the triangle $c, d, e$ is acute-angled. For if not then

$$a^2 \geq b^2 + c^2 \quad \text{and} \quad c^2 \geq d^2 + e^2;$$

and hence

$$a^2 \geq 2c^2 \geq 2(d^2 + e^2) = (d+e)^2 + (d-e)^2 \geq (d+e)^2.$$

But since $a, d, e$ form a triangle, $a < d + e$, and so the above inequality is impossible. Thus either $a, b, c$ or $c, d, e$ forms an acute-angled triangle.

**Q.928** Prove that if $x, y, z$ are real numbers and $x + y + z = 1$, then $xy + yz + zx \leq \frac{1}{3}$.

**ANS.** First we "symmetrise" the problem by letting

$$x = X + \tfrac{1}{3},\ y = Y + \tfrac{1}{3},\ z = Z + \tfrac{1}{3}.$$

Then $X + Y + Z = 0$ and we have

$$\begin{aligned} xy + yz + zx &= (XY + \tfrac{1}{3}X + \tfrac{1}{3}Y + \tfrac{1}{9}) + (YZ + \tfrac{1}{3}Y + \tfrac{1}{3}Z + \tfrac{1}{9}) \\ &\qquad + (ZX + \tfrac{1}{3}Z + \tfrac{1}{3}X + \tfrac{1}{9}) \\ &= XY + YZ + ZX + \tfrac{1}{3} \\ &= \tfrac{1}{2}\left((X+Y+Z)^2 - (X^2 + Y^2 + Z^2)\right) + \tfrac{1}{3} \\ &= \tfrac{1}{3} - \tfrac{1}{2}(X^2 + Y^2 + Z^2) \\ &\leq \tfrac{1}{3} \end{aligned}$$

as required. This also shows that for $xy + yz + zx$ to equal $\frac{1}{3}$, the only possibility is $X = Y = Z = 0$, that is, $x = y = z = \frac{1}{3}$.

**Q.929 (a)** At the upper left hand corner of an $8 \times 8$ chessboard is a counter which may be moved at most 4 squares horizontally to the right or at most 3 squares vertically downwards (not both in the same turn). Two players alternately move the counter and the winner is the first player to reach the lower right hand corner. Which player has a winning strategy?

(b) The same, except that the first player to reach the corner is the loser.

ANS. (a) We will mark the squares of the chessboard with "$W$" or "$L$" to denote that a player reaching that square will win or lose, respectively. We shall start at the lower right-hand corner, which according to the rules is a winning square, and work backwards – compare the methods used in the article "Addition Games" in issue 1 this year. If a player reaches any one of the three squares immediately above the bottom right-hand corner, the opponent can win in one move, so these three squares must be marked "$L$". On the other hand, the next square up is marked "$W$" since a player landing there forces the other player onto a losing square. The same argument shows that to the left of the corner square there are 4 squares marked "$L$" and one marked "$W$". Thus we have

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | | | | | | | |
| | | | | | | | |
| | | | | | | | |
| | | | | | | | W |
| | | | | | | | L |
| | | | | | | | L |
| | | | | | | | L |
| | | W | L | L | L | L | W |

and continuing in the same manner we get

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| L | L | L | L | W | L | L | L |
| W | L | L | L | L | W | L | L |
| L | W | L | L | L | L | W | L |
| L | L | W | L | L | L | L | W |
| L | L | L | L | W | L | L | L |
| W | L | L | L | L | W | L | L |
| L | W | L | L | L | L | W | L |
| L | L | W | L | L | L | L | W |

as the final diagram. Thus the first player wins by moving one square down or four to the right.

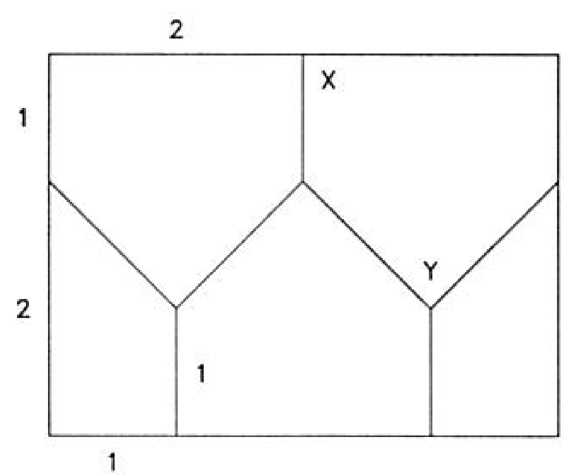
**(b)** As above, but we begin with "$L$" in the lower right-hand corner. This gives eventually

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| L | L | L | L | W | L | L | L |
| W | L | L | L | L | W | L | L |
| L | L | W | L | L | L | L | W |
| L | W | L | L | L | L | W | L |
| L | L | L | L | W | L | L | L |
| W | L | L | L | L | W | L | L |
| L | L | W | L | L | L | L | W |
| L | W | L | L | L | L | W | L |

so that the first player wins again by making the same initial moves! However the subsequent play will be different.

**Q.930** Six points are located inside, or on the boundary of, a $3 \times 4$ rectangle. Show that two of them are separated by a distance $\sqrt{5}$ or less.

**ANS.** Divide the $3 \times 4$ rectangle into five parts as shown.




If six points are located in the rectangle, at least two must be in (or on the boundary of) the same subdivision. But it is not hard to see that the maximum distance between any two points in the same section is $\sqrt{5}$ (for example, this is the distance from $X$ to $Y$). Thus two of the six points are separated by $\sqrt{5}$ or less. (In fact two points must be strictly less than $\sqrt{5}$ apart. Investigate!)

**Solutions:**

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Complete solutions to 925, 926, 927, 928, 929, 930.

Partial solutions to 923, 924.