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HUYGENS' CONSTRUCTION

FOR THE REFLECTION OF A PLANE WAVE FROM A CONCAVE PARABOLIC MIRROR

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In 1678 Christiaan Huygens developed a wave theory of light in contradiction of Newton's corpuscular theory. It was not published until 1690. **Huygens' Principle**, or "**Construction**" allows the successive shapes and positions of a travelling wavefront to be determined by using the following procedure:

- Assume that every point on a wavefront (of light) is a source of spherical (circular in two dimensions) secondary wavelets which travel through the medium at the wave speed. These wavelets are hypothetical and have an amplitude which varies as $(1+\cos\theta)$ where θ is the angle from the forward direction normal to the wavefront. (Real ripples have the same amplitude around their circumference).

- The position and shape of the wavefront at any time can be reconstructed from the envelope of all of the secondary wavelets from all of the points on an earlier wavefront or wavefronts.

As a geometrical construction, without equations, it is easy to model the propagation, reflection and refraction of plane waves in two dimensions using Huygens' Principle. The application here is rather more subtle. Consider a plane wave with wavefront perpendicular to the principal axis of a parabolic mirror at successive positions as it moves into the mirror **(Figure 1)**: As the wavefront moves in at constant velocity we can construct secondary wavelets centred on the points $P_1, P_2, \cdots, P_9$. By the time the wavefront has reached the vertex of the parabola, P_{10}, the wavelet from P_1 has a radius equal to the distance from position 1 to the vertex, that from P_2 a radius equal to the distance from position 2 to the vertex, and so on. That is the wavelets from $P_1, P_2, \cdots, P_9$ are "semi" circles of uniformly decreasing radii. The wavelets on the points on the other side of the principal axis are the mirror images of those on $P_1, P_2, \cdots, P_9$. These are all shown in Figure 2 where it is clear that the envelope of all of these wavelets is a circle centred on the focus of the parabola. (The envelope of a family of curves is a curve $\mathcal{C}$ with the property that for each point P on $\mathcal{C}$ there is a curve of the family through P and tangent to $\mathcal{C}$.)

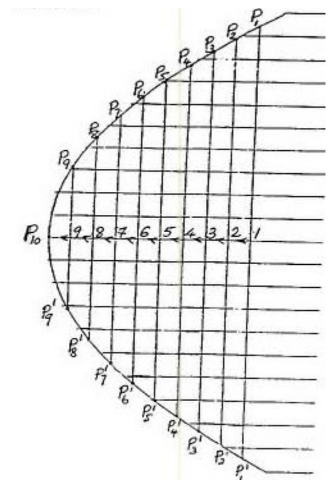

Figure 1: Successive positions of the plane wave going into the parabolic mirror

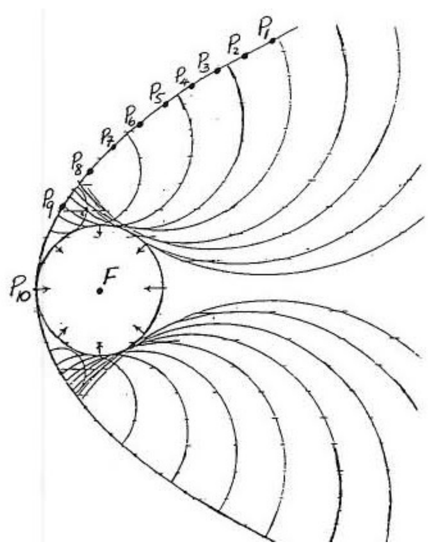

Figure 2: Secondary wavelets and their circular envelope

It is easy to imagine the wavelets from points before P_1, with successively larger radii, forming the rest of the circular envelope on the opposite side to P_{10}. Similarly the many small radius wavelets between P_9 and P_{10} complete the circle near P_{10}. The reconstructed wavefront from the reflection of a single plane wave from a parabolic mirror is a circle centred on the focus of the parabola; it is moving inwards towards the focus, the plane wave is brought to a focus at F. This is just what we observe by tracing rays and reflecting them from the parabola so that the angle of reflection equals the angle of incidence. **Analytical Proof.** Consider the parabola $y^2 = 4ax$ and the wavelet on the point $P(ap^2, 2ap)$. The secondary circular wavelet on P has radius ap^2 when the incident wavefront has reached the vertex $(0,0)$. Thus its equation is:

$$(x-ap^2)^2+(y-2ap)^2=(ap^2)^2.$$

This simplifies to a quadratic function $f(p)$ of the parameter p :

$$f(p)\equiv 2a(2a-x)p^2-4ayp+(x^2+y^2)=0. \tag{1}$$

Clearly this equation represents an infinite set of such circles, each one determined by a value of the parameter p. For any point (x,y) on the envelope there must be a value of p for which (x,y) satisfies (1) (since it touches each circle in the set at one point). Now, the partial derivative of $f(p)$ with respect to p, measures how $f(p)$ changes with variations in p. It turns out that points on the envelope must also satisfy the equation

$$\frac{\partial f}{\partial p}=f'(p)=0. \tag{2}$$

It is a little difficult to explain why this must generally be the case but because our function $f(p)$ is a quadratic we can proceed more geometrically. Since $f(p)$ is a quadratic in p, given the point (x,y), there are either $2,1$ or 0 values of p for which (1) holds; i.e. there are either $2,1$ or 0 of our secondary circular wavelets through (x,y). It is intuitively clear that any point (x,y) on the envelope belongs to exactly 1 wavelet. Therefore the roots of (1) must be equal. (Note that the roots of (1) are equal if and only if $\dfrac{\partial f}{\partial p}=0$, confirming the truth of (2) in this particular case.) Now the roots are equal when "$\triangle=b^2-4ac=0$." Looking at (1) again:

$$2a(2a-x)p^2-4ayp+(x^2+y^2)=0$$

we see $\triangle=b^2-4ac=0$ gives: (note 'a' is different in each equation).

$$\begin{aligned}(4ay)^2-4(2a(2a-x))(x^2+y^2)&=0\\ \therefore 16a^2y^2-8a(2ax^2+2ay^2-x^3-xy^2)&=0\\ \therefore 16a^2x^2-8ax^3+8axy^2&=0\end{aligned}$$

Divide through by $8ax$ to get

$$-2ax+x^2+y^2=0$$

complete the square:

$$\begin{aligned}x^2-2ax+a^2+y^2&=a^2\\ \therefore (x-a)^2+y^2&=a^2.\end{aligned} \tag{3}$$

This is the equation of the envelope of all of the secondary wavelets, a circle of radius a centred on the focus $(a,0)$ – wow!!