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HERON'S FORMULA AND SOME MYSTERIOUS TRIANGLES

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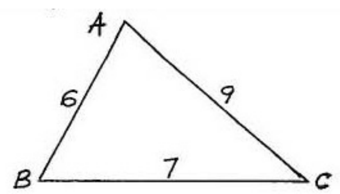
Suppose your teacher asks you to find the area of the following triangle. Your approach



would probably be to use the cosine rule to find, say, angle C and then use the formula $A = \frac{1}{2}ab\sin C$ to get the area. Specifically:

$$\cos C = \frac{7^2 + 9^2 - 6^2}{2.7.9} = \frac{47}{63}$$
$$so \qquad C \simeq 41.75^\circ.$$

Hence the area is approximately 20.975 units^2. In fact, the exact area is $2\sqrt{110}$ units^2. One way to see this is to write $A = \frac{1}{2}ab\sqrt{1 - \cos^2 C}$

$$= \frac{1}{2}.7.9\sqrt{1 - (\frac{47}{63})^2} = 2\sqrt{110},$$

but you can get this answer directly **without** finding any of the angles in the triangle. Given any triangle, with sides a, b, c, we calculate the quantity s (called the **semiperimeter**) given by

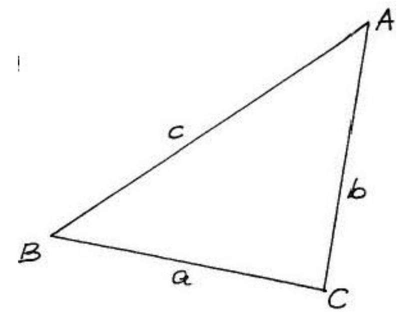
$$s = \frac{1}{2}(a + b + c).$$

The ancient Greek mathematician Heron (circa AD62) showed that the area of the triangle is given by

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

(Probably Heron's proof was due to Archimedes but nevertheless the formula is now known as Heron's formula.) We can easily establish the result using the ideas we started with and some pretty applications of the difference of two squares.

Starting with triangle ABC



we can write $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$ and use the area formula

$$\begin{aligned} A &= \frac{1}{2}ab\sqrt{1 - \cos^2 C} \\ &= \frac{1}{2}ab\sqrt{1 - \frac{(a^2 + b^2 - c^2)^2}{4a^2b^2}} \\ &= \frac{1}{4}\sqrt{4a^2b^2 - (a^2 + b^2 - c^2)^2} \end{aligned}$$

Now, by the way, as a general principle in mathematics you should never expand a complicated algebraic expression unless someone is holding a gun to your head (or alternatively you've exhausted all other modes of attack)!

The expression under the square root sign is a difference of two squares, so

$$A = \frac{1}{4}\sqrt{[2ab - (a^2 + b^2 - c^2)][2ab + (a^2 + b^2 - c^2)]}$$

Once again, expanding is disastrous, but you should recognise a couple of perfect squares in the brackets, giving

$$A = \frac{1}{4}\sqrt{[c^2 - (a - b)^2][(a^2 + b)^2 - c^2]}$$

and another difference of two squares gives

$$A = \frac{1}{4}\sqrt{(c - a + b)(c + a - b)(a + b - c)(a + b + c)}.$$

Finally, taking the $\frac{1}{4}$ inside the square root sign and 'spreading it around', we get

$$A = \sqrt{(\frac{c - a + b}{2})(\frac{c + a - b}{2})(\frac{a + b - c}{2})(\frac{a + b + c}{2})}$$

which becomes more compactly,

$$A = \sqrt{s(s - a)(s - b)(s - c)}$$

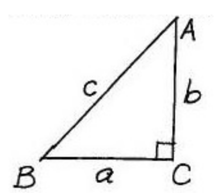
when we introduce the semiperimeter s. By the way, this was not the proof published by Heron. He used some complicated geometry, since he didn't have trigonometry at his disposal. **Example:** Find the area of the triangle with sides 8, 6 and 4. Here $s = 9$, so the area A is given by

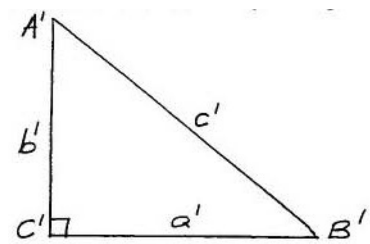
$$A = \sqrt{9.1.3.5} = 3\sqrt{15}.$$

This is surely much quicker than using trigonometry! (AND you don't need a calculator!) **Question.** Can you find a non-right angled triangle with integer side lengths and integer area?

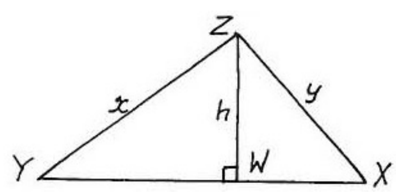
There are in fact infinitely many such triangles. For example a triangle of sides 13, 14, 15 has area 84 units^2. Such triangles are called Heron triangles.

There are several algorithms for finding examples of Heron triangles. One simple method is to take two Pythagorean triads, (a, b, c), (a', b', c'), and 'glue' together the corresponding triangles in some way.





Let $h = lcm(b, b')$ i.e., the lowest common multiple of b and b'. Then $h = bn = b'n'$. Construct triangles ZYW and ZXW similar to triangles ABC and $A'B'C'$ respectively



with $h = ZW$. Then by similar triangles, we have

$$x = nc, \; y = n'c', \; z = na + n'a'.$$

Now $\triangle XYZ$ is a Heron triangle, since its area is $\frac{1}{2}h(na + n'a')$ which is always an integer (why?)

To illustrate this result, choose the Pythagorean triads $(3, 4, 5)$, $(5, 12, 13)$.

$$h = lcm(4, 12) = 12, \quad \text{so} \quad n = 3, \; n' = 1$$

whence $x = 15$, $y = 13$, $z = 9 + 5 = 14$ and $(15, \; 13, \; 14)$ is Heron triangle of area 84 as we stated above.

A more complicated algorithm was given by the Indian mathematician Brahmagupta in the 7th century AD, but since it would take me too long to explain why it works, I will simply give you the result.

Choose any two positive rational numbers α and β such that $\alpha\beta$ is greater than 1 and let $\gamma = \frac{\alpha+\beta}{\alpha\beta-1}$. Write α, β, γ with a common denominator and take the numerators α', β', γ'. Then with $x = \alpha' + \beta', \ y = \alpha' + \gamma', \ z = \beta' + \gamma'$, we have the sides x, y, z of a Heron triangle. For example, take $\alpha = \frac{4}{3}, \ \beta = \frac{7}{5}$ then $\gamma = \frac{41}{13}$. Putting these over a common denominator we get

$$\frac{260}{195}, \frac{273}{195}, \frac{615}{195}, \text{ so } (\alpha', \beta', \gamma') = (260, 273, 615)$$

$$\text{so} \quad (x, y, z) = (533, 875, 888)$$

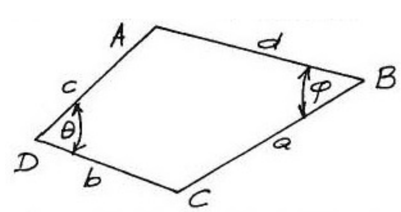
which is a Heron triangle with area 223860. Interestingly, if we choose α, β such that $(\alpha - 1)(\beta - 1) = 2$ then $\gamma = 1$ and a little algebra will reveal the fact that (y, z, x) is then a Pythagorean triad!

Heron listed a number of Heron triangles among which were many examples with **consecutive** integer sides. While he seems to have believed that there were infinitely many such triangles with integer areas, he could not give any systematic way of finding all of them. This is not surprising, since the answer comes down to a problem in number theory which was not solved till much later.

If $n - 1, \ n, \ n + 1$ are the sides of a triangle with integer area then we can find **all** the integer values of n by taking n to be the nearest whole number to $(2 + \sqrt{3})^k, k = 1, 2, 3, \ldots,$. Not surprisingly this requires some fairly 'high powered' mathematics (in particular continued fractions). The area is given by $\frac{1}{4}n\sqrt{3n^2 - 12}$ (this is easy to show) which will be (surprisingly) an integer when n is taken as above. Here are the first few

k	n-1	n	n+1	Area
1	3	4	5	12
2	13	14	15	84
3	51	52	53	1170
4	193	194	195	16296

Triangles are very nice objects, in that they have enough rigid structure to be able to say a lot about them. The same is not quite true for quadrilaterals. For example if we only know the sides of a quadrilateral then the area is not uniquely determined. We need to know also, two of the angles in the quadrilateral.



Heron's formula can then be generalised slightly in the following way.

Let $s = \dfrac{a+b+c+d}{2}$ and $\alpha = \dfrac{\theta+\varphi}{2}$.
Then the area is given by

$$A = \sqrt{(s-a)(s-b)(s-c)(s-d) - abcd\cos^2\alpha}.$$

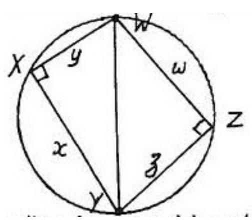
This formula is harder to prove, although the proof only uses 'basic' trigonometry and algebra. Moreover it is not as elegant or simple as Heron's formula.

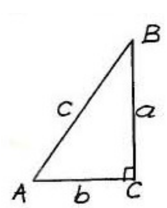
However, if we have a **cyclic** quadrilateral, then the opposite angles are supplementary. Hence $\theta + \varphi = 180°$ and so $\alpha = 90°$. In this case the formula collapses to

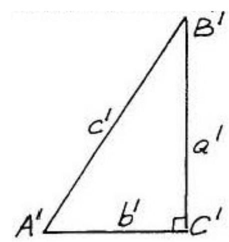
$$A = \sqrt{(s-a)(s-b)(s-c)(s-d)}$$

which is sometimes called Brahmagupta's Theorem (Brahmagupta seems to have thought that the formula was true for all quadrilaterals.)

The problem of finding cyclic quadrilaterals with integer sides and integer area is somewhat more difficult, but one way of generating some of them might be to take two Pythagorean triads and glue them together in a similar manner to that which we used above. Since two of the opposite angles are $90°$, the resulting quadrilateral will be cyclic.







Let $h = lcm(c, c')$ so $h = nc$ and $h = n'c'$.

Let $\triangle WYX$ and $\triangle WYZ$ be similar to triangles ABC and $A'B'C'$ respectively with $WY = h$, then by similar triangles, $x = na,\ y = nb,\ z = na',\ w = nb'$ then x, y, z, w are the sides of a cyclic quadrilateral with integer area.

For example, if we take Pythagorean triads $(3, 4, 5),\ (5, 12, 13),\ h = lcm(5, 13) = 65$, so $n = 123,\ n' = 5$ and $(39,\ 52,\ 25,\ 60)$ are the desired side lengths with area, 1764.

Can you find a cyclic quadrilateral with consecutive integer sides and integer area? I leave this with you as a challenge problem to either find such a quadrilateral or prove that no such quadrilateral exists.