Parabola Volume 31, Issue 1 (1995)

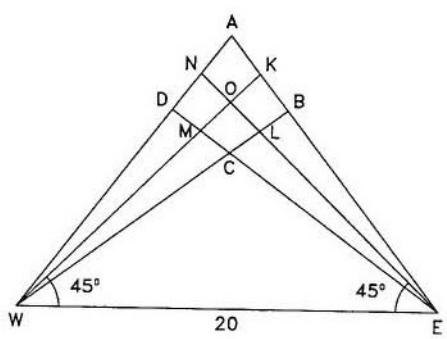
SOLUTIONS TO PROBLEMS 931-940

Q.931 In a large flat area of bushland there are two fire-spotting towers, one exactly 20km east of the other. A bushfire is reported as being due north-east of the western tower, and simultaneously due north-west of the eastern tower. However each of these directions could be in error by up to 1° either way. Find the total area within which the fire might be located

a. by a simple approximate argument;

b. exactly.

ANS. Consider the following diagram (in which the 1° angles have been magnified for clarity). The fire must be located within $ABCD$.



(a) Since $\angle DAB = 88^\circ$, $\angle BCD = 92^\circ$ and $\angle ABC = \angle CDA = 90^\circ$, $ABCD$ is very nearly a square. Also NL is almost a 2° arc of a circle with radius $OW = 20/\sqrt{2} = 10\sqrt{2}$. Hence

$$\text{area } ABCD \simeq NL^2 \simeq \left(\frac{2}{360} \times 2\pi \times 10\sqrt{2}\right)^2 = \frac{2\pi^2}{81}.$$

(b) To find the exact area, recall that $\angle ABC = \angle CDA = 90^\circ$, and note that $AB = AD$ and $BC = DC$. Thus $ABCD$ can be divided into two congruent right-angled triangles $\triangle ABC$ and $\triangle ADC$, and

$$\text{area } ABCD = 2 \times \tfrac{1}{2}(AB)(BC) = (AB)(BC).$$

Clearly the four angles at O are all right angles. Hence

$$\begin{aligned} AB &= WB \tan 2^\circ \\ &= WK \cos 1^\circ \tan 2^\circ \\ &= (WO + OK) \cos 1^\circ \tan 2^\circ \\ &= (WO + EO \tan 1^\circ) \cos 1^\circ \tan 2^\circ \\ &= 10\sqrt{2}\,(1 + \tan 1^\circ) \cos 1^\circ \tan 2^\circ; \end{aligned}$$

similarly

$$\begin{aligned} BC &= (EO - OL)\cos 1^\circ \tan 2^\circ \\ &= 10\sqrt{2}\,(1 - \tan 1^\circ)\cos 1^\circ \tan 2^\circ \end{aligned}$$

and the area is

$$\begin{aligned} 200(1 - \tan^2 1^\circ)\cos^2 1^\circ \tan^2 2^\circ &= 200(\cos^2 1^\circ - \sin^2 1^\circ)\tan^2 2^\circ \\ &= 200\cos 2^\circ \tan^2 2^\circ. \end{aligned}$$

(Check: to six decimal places, this area is $0.243743\ \text{km}^2$, while the approximation in (a) is $0.243694\ \text{km}^2$.)

Q.932 Four weary explorers have to cross a bridge over a river one night. Owing to their various degrees of exhaustion, they would individually take 5, 10, 20 and 25 minutes (respectively) to cross the bridge. However, the old and rickety bridge will take only one or two people at a time. Furthermore, it is too dangerous to cross the bridge in the dark, and the expedition has only one torch. How can all four explorers cross the bridge in the least possible total time?

ANS. The fastest scheme is as follows (by "5" I mean "the person who can cross in 5 minutes", and so on):
5 and 10 cross the bridge .... 10 minutes
5 returns with torch .... 5 minutes
20 and 25 cross .... 25 minutes
10 returns with torch .... 10 minutes
5 and 10 cross .... 10 minutes
Total, 60 minutes.

To confirm that the task cannot be accomplished in less than 60 minutes, note that at least eight journeys must be made (three crossings by two explorers, and two returns by one explorer); and that each person must make an odd number of trips. It is clear that to complete the transfer in under 60 minutes, 25 and 20 must make only one trip each. If they make separate trips, these will take 25 and 20 minutes respectively; there must be one other trip involving two explorers and taking 10 minutes; and two additional trips taking (at least) 5 minutes each. The total time would be at least

$$25 + 20 + 10 + 5 + 5 = 65 \text{ minutes},$$

which is too long. Therefore 25 and 20 must cross together. Now there are to be three crossings involving two people, so 10 must cross twice and return once; and the fifth trip must consist of 5 returning alone. So the minimum total time for the transfer is

$$25 + 10 + 10 + 10 + 5 = 60 \text{ minutes},$$

as claimed.

Q.933

a. In how many ways can 1994 be written as the sum of (one or more) consecutive positive integers?

b. Prove that for any positive integer n, the number of ways of writing n as the sum of (one or more) consecutive positive integers is equal to the number of odd factors of n.

c. Deduce from (b) that a positive integer can be written as the sum of two or more consecutive positive integers if and only if it is not a power of 2.

ANS. We solve (b) first. If n is the sum of k consecutive positive integers starting at a, we have
$$n = a + (a+1) + \cdots + (a+k-1) = \tfrac{1}{2}k(2a+k-1)$$
and so
$$2n = k(2a+k-1). \qquad (*)$$
Given n, we wish to find a and k. Note that the difference between k and $2a+k-1$ is $2a-1$, an odd number; so one of the factors in $(*)$ is odd and the other even. Moreover, $k < 2a+k-1$. Thus all solutions of $(*)$ are given as follows: let d be an odd factor of $2n$; then $2n/d$ is even. If $d < 2n/d$ solve
$$k = d,\ 2a+k-1 = 2n/d$$
to find a and k; if $d > 2n/d$ solve
$$k = 2n/d,\ 2a+k-1 = d.$$
For every odd factor of $2n$ we find exactly one solution. But an odd number is a factor of $2n$ if and only if it is a factor of n. Therefore the number of solutions is the number of odd factors of $2n$, which is the same as the number of odd factors of n.

In the particular case $n = 1994 = 2 \times 997$, there are only two odd factors of n, namely 1 and 997. The corresponding values of a and k are given by
$$k = 1,\ 2a+k-1 = 3988 \text{ so } a = 1994$$
$$k = 4,\ 2a+k = 1 = 997 \text{ so } a = 497.$$
Thus we have
$$1994 = 1994 = 497 + 498 + 499 + 500,$$
a sum of one or four consecutive integers.

To solve (c), note that an integer can always be written as the "sum" of one consecutive integer (namely, itself), and, from above, it can therefore be written as the sum of two or more consecutive positive integers if and only if it has at least two odd factors. But every integer has 1 as an odd factor, and only powers of 2 have no other odd factors.

Q.934 Prove that there is no polyhedron (solid figure bounded by plane surfaces) having 7 edges, but there is one with any number of edges greater than 7.

ANS. Suppose that there is a polyhedron with seven edges. Let $f_3, f_4, \ldots$ be the number of faces of the polyhedron which are triangles, quadrilaterals, $\ldots$ respectively; and let $v_3, v_4, \ldots$ be the number of vertices at which respectively $3, 4, \ldots$ edges meet. Now we can count the total number of edges by counting three for every triangular face, four for every quadrilateral and so on; but we must remember that every edge will be counted twice by this method. So

$$3f_3 + 4f_4 + 5f_5 + \cdots = 14. \tag{1}$$

Similarly, considering the vertices yields the equation

$$3v_3 + 4v_4 + 5v_3 + \cdots = 14. \tag{2}$$

Also, Euler's well-known formula $F + V = E + 2$ becomes in this case

$$f_3 + v_3 + f_4 + v_4 + f_5 + v_5 + \cdots = 9. \tag{3}$$

Taking equation (1) plus (2) minus three times (3),

$$(f_4 + v_4) + 2(f_5 + v_5) + \cdots = 1.$$

But note that none of the terms on the LHS may be negative; hence

$$f_5 = v_5 = f_6 = v_6 = \cdots = 0$$

and either

$$f_4 = 1,\ v_4 = 0$$

or

$$f_4 = 0, v_4 = 1.$$

Subtracting (1) from (2) and rearranging,

$$3(f_3 - v_3) = -4(f_4 - v_4) = \pm 4$$

which is impossible as $f_3 - v_3$ is an integer.

To show that any number of edges greater then 7 is possible, consider a square pyramid, a triangular prism and a pentagonal pyramid: these have 8,9 and 10 edges respectively, and each has at least one triangular face. On one triangular face of each solid construct a shallow triangular pyramid: this gives three extra edges, thus creating polyhedra with 11, 12 and 13 edges. Each of these still has a triangular face, so the procedure may be repeated to give any number of edges greater than 7. (Comment: there is also a polyhedron with 6 edges, namely, a triangular pyramid.)

Q.935 A triangle has integer sides. Each side is divided into intervals of length 1 and the midpoint of each interval is marked. Prove that it is possible to draw a continuous path, linking all these midpoints and returning to its starting point, subject to the following conditions:

a. every point is visited once, and no point is used more than once (except that the first point is the same as the last);

b. successive points on the path must come from different sides of the triangle.

ANS. Let the lengths of the three sides be $a \geq b \geq c$; note that $a \leq b + c$. Write $k = b + c - a \geq 0$. Then the path formed by visiting the sides in the order

$$\overbrace{abc \cdots abc}^{k \text{ triples}} \; \overbrace{ab \cdots ab}^{b-k \text{ pairs}} \; \overbrace{ac \cdots ac}^{c-k \text{ pairs}} \; a$$

satisfies the requirements of the question.

Q.936 Find all solutions in positive integers of

$$6x^2 + 3y^2 + 6z^2 - 8xy - 8yz + 10xy = 6.$$

ANS. Rearranging the equation,

$$(2x - y + z)^2 + (x - y + 2z)^2 + (x - y + z)^2 = 6.$$

Now the only way to write 6 as the sum of three squares is

$$6 = 2^2 + 1^2 + 1^2.$$

Thus $2x - y + z,\ x - y + 2z,\ x - y + z$ must be $\pm 2,\ \pm 1,\ \pm 1$ in some order. Now the equations

$$\begin{aligned} 2x - y + \ \ z &= a \\ x - y + 2z &= b \\ x - y + \ \ z &= c \end{aligned}$$

can be solved without much difficulty to give

$$x = a - c,\ y = a + b - 3c,\ z = b - c.$$

Noting that $x - y + z$ is strictly smaller than the other two expressions, the possible values for (a, b, c) are

$$(2, 1, -1),\ (1, 2, -1),\ (1, -1, -2),\ (-1, 1, -2),$$

which yield the following four solutions for (x, y, z) :

$$(3, 6, 2),\ (2, 6, 3),\ (3, 6, 1),\ (1, 6, 3).$$

Q.937 An $n \times n$ chessboard has a number of beans placed on each square. The squares in the top row contain (from left to right) $1, 2, 3, \ldots, n$ beans; in the second row $n + 1,\ n+2,\ n+3, \ldots, 2n$; and so on, ending with n^2 beans in the bottom right hand corner. It is permitted to select any two rows and remove from each square in one of them the number of beans in the corresponding square in the other one. For example, if one row contains 1,4,3 beans and another 2,7,4 the latter may be changed to 1,3,1; then in the next move, the first row may become 0,1,2. “Negative beans” are not allowed (for example, no change is possible on the above rows containing 0,1,2 and 1,3,1 beans respectively).

a. After performing the above operation as many times as you wish, what is the smallest possible remaining number of non-empty rows on the chessboard?

b. What is the minimum possible total number of remaining beans?

c. What are the answers to (a) and (b) if we allow not only the above operation on the rows of the chessboard, but also a similar operation on the columns?

ANS.(a) The board can be reduced to two non-empty rows as follows. First subtract row $n-1$ from row n; then row $n-2$ from row $n-1$; and so on, finally subtracting row 1 from row 2. This leaves the following situation:

1	2	3	$\cdots$	n
n	n	n	$\cdots$	n
n	n	n	$\cdots$	n
$\vdots$	$\vdots$	$\vdots$		$\vdots$
n	n	n	$\cdots$	n

Now subtracting the second row from each subsequent row leaves only the first two rows non-empty. It is not possible to reduce the board to a single row; for if it were so, the original configuration must have consisted entirely of multiples of this row, which is clearly not the case.
(b) From the situation above consisting of rows of $(1, 2, 3, \ldots, n)$ and $(n, n, n, \ldots, n)$ beans, subtract the first from the second to give

$$(1, 2, 3, \ldots, n) \text{ and } (n-1, n-2, n-3, \ldots, 0),$$

a total of n^2 beans. We shall prove that this is the minimum possible. In what follows, to add or subtract two rows means to add or subtract the numbers in corresponding squares:

$$(a_1, a_2, \ldots, a_n) \pm (b_1, b_2, \ldots, b_n) = (a_1 \pm b_1, a_2 \pm b_2, \ldots, a_n \pm b_n);$$

and to multiply a row by a constant means to multiply each number by that constant:

$$s(a_1, a_2, \ldots, a_n) = (sa_1, sa_2, \ldots, sa_n).$$

First note that every row initially can be written in terms of two possible rows: row k is

$$\begin{aligned}&(kn-(n-1), k_n = (n-2),\ k_n-(n-3), \ldots, k_n)\\ &\quad = s(1, 2, 3, \ldots, n) + t(n-1, n-2, n-3, \ldots, 0)\end{aligned}$$

where $s = k,\ t = k - 1$. On subtracting one row from another we obtain a row

$$[s(1,2,3,\dots,n) + t(n-1,n-2,n-3,\dots,0)] \\ - [s'(1,2,3,\dots,n) + t'(n-1,n-2,n-3,\dots,0)],$$

that is,

$$(s-s')(1,2,3,\dots,n) + (t-t')(n-1,n-2,n-3,\dots,0),$$

which is still expressed in terms of the two basic rows. Now by repeating row-subtractions we can never entirely eliminate either of these rows; for if so, then every row would initially have been a multiple of one row, which, as we saw above, is impossible. Thus the minimum total number of beans consists of one of each basic row, that is, n^2 beans.
(c) By using subtraction of columns as well as rows we cannot reduce the board to fewer than two rows. First note that a row-subtraction (say, row k minus row k') and a column-subraction (say, column ℓ minus column ℓ') will give the same result irrespective of which occurs first. This is because any square on the board other than that in row k, column ℓ is altered by only one (or neither) of these altered operations; and if row k, column ℓ contains c beans, if row k, columnn ℓ' contains a and if row k', column ℓ contains b, then performing the row-subtraction first leaves

$$(c-b)-a$$

beans in row k, column ℓ, while performing the comumn-subtraction first leaves

$$(c-a)-b,$$

which is the same. Now since row and column-subtractions may be interchanged, we can assume that all the row-operations take place first, leaving at least two non-empty rows, and then the column-operations. But a row cannot be entirely emptied by column-operations, so there must still remain two non-empty rows.

We can, however, reduce the total number of beans. First, as in (a), reduce by row-subtractions to

$$(1,2,3,\dots,n) \text{ and } (n,n,n,\dots,n),$$

with all other rows empty. Then subtract column $n-1$ from column n, column $n-2$ from column $n-1, \dots$, column 1 from column 2, leaving

$$(1,1,1,\dots,1) \text{ and } (n,0,0,\dots,0).$$

Finally subtract column n from all others, giving

$$(0,0,0,\dots,1) \text{ and } (n,0,0,\dots,0),$$

a total of $n+1$ beans.

Q.938 Let α be a constant, not a multiple of π. Show that the x-axis is tangent to the curve

$$y = x - \sin x - (1-\cos x)\tan\alpha$$

at the origin; and that it is also tangent elsewhere if and only if $\tan\alpha - \alpha$ is a multiple of π.

ANS. Note that the question as given was slightly incorrect; a revised version appears above. The x-axis is tangent to the curve if and only if there is a point x at which y and $\frac{dy}{dx}$ are simultaneously zero, that is,

$$x - \sin x - (1 - \cos x)\tan\alpha = 0. \tag{1}$$

$$1 - \cos x - \sin x \tan\alpha = 0. \tag{2}$$

It is easy to see that if $x = 0$ these equations are both true, so the x-axis is tangent to the curve at the origin. Now suppose there is a solution $x \neq 0$. Rearrange (2) as

$$\cos x = 1 - \sin x \tan\alpha, \tag{3}$$

square both sides,

$$\cos^2 x = 1 - 2\sin x \tan\alpha + \sin^2 x \tan^2\alpha,$$

subtract $\cos^2 x$ from each side and write $\sin^2 x$ instead of $1 - \cos^2 x$,

$$\sin^2 x - 2\sin x \tan\alpha + \sin^2 x \tan^2\alpha = 0,$$

factorise and write $\sec^2\alpha$ for $1 + \tan^2\alpha$:

$$\sin x\,(\sin x \sec^2\alpha - 2\tan\alpha) = 0.$$

If $\sin x = 0$ then (1) and (2) become

$$x - (1 - \cos x)\tan\alpha = 0, \quad 1 - \cos x = 0,$$

so $x = 0$. But this is the case we have excluded. Therefore we must have $\sin x \sec^2\alpha - 2\tan\alpha = 0$, that is,

$$\sin x = 2\tan\alpha\cos^2\alpha = 2\sin\alpha\cos\alpha = \sin 2\alpha. \tag{4}$$

Substituting in (3),

$$\cos x = 1 - 2\sin\alpha\cos\alpha\tan\alpha = 1 - 2\sin^2\alpha = \cos 2\alpha. \tag{5}$$

From (4) and (5),

$$x = 2\alpha + 2n\pi$$

for some integer n. On the other hand, substituting (4) and (5) back into (1) gives

$$x = 2\sin\alpha\cos\alpha + 2\sin^2\alpha\tan\alpha = 2\tan\alpha(\cos^2\alpha + \sin^2\alpha) = 2\tan\alpha$$

and so we have

$$2\tan\alpha - 2\alpha = 2n\pi,$$

that is, $\tan\alpha - \alpha = n\pi$, a multiple of π, as claimed.

Conversely, if $\tan\alpha - \alpha = n\pi$ then choose

$$x = 2\tan\alpha = 2\alpha + 2n\pi.$$

Then

$$\begin{aligned} \sin x &= \sin 2\alpha = 2\sin\alpha\cos\alpha \\ 1-\cos x &= 1-\cos 2\alpha = 2\sin^2\alpha \end{aligned}$$

and it is now easy to check that (1) and (2) hold.

Q.939 Which of the statements in the following list are true?

1. At least one odd-numbered statement in this list is false.
2. Either the second or third statement in this list is true.
3. This list does not contain two consecutive false statements.
4. There are at least two false statements in this list.
5. If the first statement in this list is deleted, the number of true statements will decrease.

ANS. Before we can determine whether statement 5 is true or false we must consider what happens when the first statement is deleted: that is, consider statements 2,3,4 and 5 only (in that order).

Suppose statement 4 is true. Since it is the third statement in this list, 2 is also true. Now 4 says (truly) that the list contains (at least) two false statements, so 3 and 5 are false; but then the list does not contain two consecutive false statements, so 3 is true after all. This is impossible; so statement 4, which we assumed to be true, must in fact be false. This being so, the list contains only one false statement (namely, statement 4 itself) and three true statements.

Now we return to the complete list of five statements. The fifth says that the list contains four or five true statements. If this is true then there is only one false statement, or none at all, in the list; so 3 must be true. Now consider statement 1. Since we know that 3 and 5 are true, 1 says (in effect) "this statement is false". Such a statement can be neither true nor false, so we have an impossible situation. Thus statement 5 must, after all, be false. It follows immediately that 1 is true. If statement 4 is false then (since we know 5 is false) there are at least two false statements, and so 4 is in fact true. (Summary: so far we know that 5 is false while 1 and 4 are true.) Finally, if statement 3 is true then so is 2; but this is impossible as there would be four true statements, making statement 5 true. Therefore statement 3 is false, and to get two consecutive false statements in the list, 2 must also be false.
Answer: statements 1 and 4 are true, and statements 2, 3 and 5 are false.

Q.940 Write a polynomial $p(x)$ in the following form:

$$p(x) = a_0 + b_0x + c_0x^2 + d_0x^3 + a_1x^4 + b_1x^5 + c_1x^6 + d_1x^7 + a_2x^8 + \cdots ,$$

where all the coefficients are real numbers. Show that $p(x)$ is divisible by x^2+1 if and only if

$$a_0 + a_1 + \cdots = c_0 + c_1 + \cdots \quad \text{and} \quad b_0 + b_1 + \cdots = d_0 + d_1 + \cdots .$$

ANS. First note that the polynomial $y+1$ is a factor of y^k+1 for odd k, and of y^k-1 for even k. Replacing y by x^2, we see that x^2+1 is a factor of $x^6+1,\ x^{10}+1,\ x^{14}+1,\ldots$ and also of $x^4-1,\ x^8-1,\ x^{12}-1,\ldots$. Therefore

$$\begin{aligned} x^2 &= x^2+1-1 = M_2 - 1 \\ x^4 &= x^4-1+1 = M_4 + 1 \\ x^6 &= x^6+1-1 = M_6 - 1 \\ x^8 &= x^8-1+1 = M_8 + 1 \end{aligned}$$

and so on, where each M_k is a multiple of x^2+1. Hence

$$\begin{aligned} p(x) &= (a_0 + c_0x^2 + a_1x^4 + c_1x^6 + a_2x^8 + \cdots) \\ &\qquad + x(b_0 + d_0x^2 + b_1x^4 + d_1x^6 + \cdots) \\ &= (a_0 + c_0(M_2-1) + a_1(M_4+1) + c_1(M_6-1) + a_2(M_8+1) + \cdots) \\ &\qquad + x(b_0 + d_0(M_2-1) + b_1(M_4+1) + d_1(M_6-1) + \cdots) \\ &= (a_0 - c_0 + a_1 - c_1 + a_2 - \cdots) + x(b_0 - d_0 + b_1 - d_1 + \cdots) \\ &\qquad + (c_0M_2 + a_1M_4 + c_1M_6 + a_2M_8 + \cdots) \\ &\qquad + x(d_0M_2 + b_1M_4 + d_1M_6 + \cdots). \end{aligned}$$

Divide $p(x)$ by x^2+1. Since every M_k is divisible exactly by x^2+1, the remainder is

$$(a_0 - c_0 + a_1 - c_1 + a_2 - \cdots) + x(b_0 - d_0 + b_1 - d_1 + \cdots).$$

Thus $p(x)$ is divisible by x^2+1 if and only if this remainder is identically zero, that is,

$$a_0 + a_1 + \cdots = c_0 + c_1 + \cdots \text{ and } b_0 + b_1 + \cdots = d_0 + d_1 + \cdots .$$

Alternative solution (for those familiar with complex numbers). Since $p(x)$ has real coefficients,

$$\begin{aligned} & p(x) \text{ is divisible by } x^2+1 \\ \Leftrightarrow\quad & p(x) \text{ is divisible by } x - i \\ \Leftrightarrow\quad & p(i) = 0 \\ \Leftrightarrow\quad & a_0 + b_0i - c_0 - d_0i + a_1 + b_1i - c_1 - d_1i + a_2 + \cdots = 0 \\ \Leftrightarrow\quad & a_0 - c_0 + a_1 - c_1 + a_2 + \cdots = 0 \text{ and } b_0 - d_0 + b_1 - d_1 + \cdots = 0 \end{aligned}$$

which is the required result.