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A PURELY GEOMETRICAL PROOF OF HERON'S FORMULA

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The famous formula of Heron connects the sides, a, b, c of a triangle with the area, A of the triangle, i.e.

$$A = \sqrt{s(s-a)(s-b)(s-c)},$$

where

$$s = \frac{a+b+c}{2}.$$

In a recent issue of **Parabola** we have seen a largely algebraic proof of this formula ("Heron's formula and some mysterious triangles" by Peter Brown.) In this article I am going to show a purely geometrical proof of it. We will connect it with properties of the inscribed and escribed circles which are worth while studying for their own interest as well.

It is well-known that every triangle has an inscribed circle, i.e. a circle which is tangent to all 3 sides of the triangle. We shall denote its centre, the incentre, by I, its radius, the inradius by ρ. Given the triangle, ABC, we can find I, by bisecting the 3 angles of the triangle. These 3 bisectors are concurrent at I. The proof of this fact is not difficult, but it uses some ideas that keep on recurring in geometry. Take any point P on the bisector of angle A and drop perpendiculars from P to AB and AC. The lengths of these perpendiculars represent the distances of P from AB, respectively AC. It is easy to prove, using congruent triangles, that these two distances are equal to each other. Furthermore, if P is a point not on the bisector of angle A, then its distances from AB and AC are different in length. This fact is expressed briefly by the statement that the anglebisector is the locus of the points equidistant from the two sides of the angle. Similarly, the bisector of angle C is the locus of points equidistant from CA and CB. Let these two bisectors meet at I. Then I is a point equidistant from AB and AC as well as from AC and BC, consequently I is a point on the bisector of angle B as well. The 3 perpendiculars from I to the sides of the triangle are equal in length, call it ρ, then the circle drawn with I as centre and ρ as radius will be tangent to the 3 sides. This is the incircle.

Now extend AC to $\overline{Y}$ and AB to $\overline{Z}$ (figure 1) and bisect $\angle\overline{Y}CB$. Let this anglebisector meet the extended line AI in E_1. Then, using again the locus property of the anglebisectors, we may conclude that E_1 is equidistant from $C\overline{Y}$, CB and $B\overline{Z}$, so it is the centre of a circle which is tangent to the 3 sides of the triangle externally. Let

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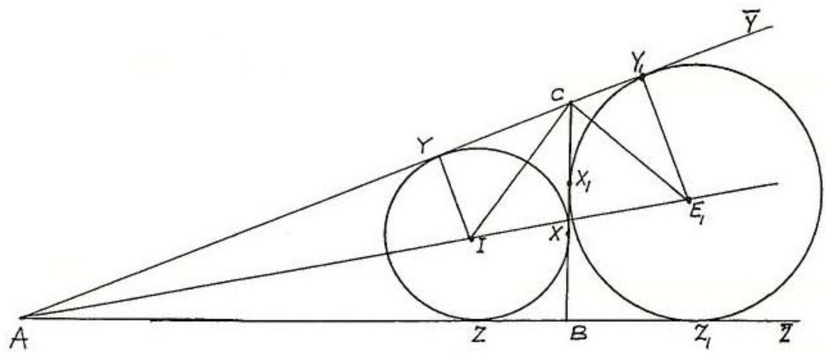

Figure 1

the radius of this circle be ρ_1. There are 3 such "excircles" or escribed circles, each being tangent to one of the sides of the triangle as well as to the extensions of the other two sides. It is a rewarding experience to the reader to construct a fair-sized general triangle, together with its incircle and the 3 excircles.

There is an easily established relation between ρ and the area of the triangle. Join I to each vertex, A, B, C this divides the triangle into 3 smaller $\triangle$'s, each with altitude δ, base being a side of the triangle, so

$$A = \rho \frac{a+b+c}{2} = \rho s \tag{1}$$

Similarly, we can see that

$$A = \rho_1 \left(\frac{b+c-a}{2} \right) = \rho_1 (s-a) \tag{2}$$

Multiplying (1) by (2), we get

$$A^2 = \rho \rho_1 s(s-a). \tag{3}$$

Theorem I *Let the incircle be tangent to BC at X, to AC at Y and to AB at Z. Then:*

$$\begin{aligned} AY &= AZ = s-a, \\ BZ &= BX = s-b, \\ CY &= CX = s-c. \end{aligned}$$

Theorem II *Let the excircle drawn around E_1 touch BC at X_1, $C\overline{Y}$ at Y_1 and $B\overline{Z}$ at Z_1. Then*

$$\begin{aligned} AY_1 &= AZ_1 = s, \\ CY &= CX_1 = s-b, \\ BZ_1 &= BX_1 = s-c. \end{aligned}$$

We left the proofs of these theorems as a challenge to the reader. They appear as exercises at the end of our article.

Now let us return to Heron's formula. Consider the triangles $\triangle CIY$ and $\triangle E_1CY_1$. Both are rightangled (at Y, respectively Y_1). We are going to prove that they are similar triangles, i.e. their remaining two angles are also equal to each other. CI and CE_1 are bisectors of the internal and external angles at C, which add up to 180°, so $\angle ICE_1 = 90^\circ$.

Therefore

$$\angle Y_1CE = 90^\circ - \angle YCI = \angle YIC.$$

Corresponding sides of similar triangles are proportional to each other, so

$$YI \div YC = CY_1 \div Y_1E_1.$$

Using the results of Theorem I and II,

$$\rho \div (s-c) = (s-b) : \rho_1,$$

or

$$\rho \times \rho_1 = (s-b)(s-c) \tag{4}$$

Substituting this into equation (3) we get:

$$A^2 = s(s-a)(s-b)(s-c),$$

which is Heron's formula.

Exercises:

1. Prove Theorems I and II from the article about Heron's formula.
2. Given the lengths of $\rho,\ \rho_1$ and BC, describe how to construct $\triangle ABC$, using only a compass and a straightedge.

Solutions:

1. Theorem I, following the notation of Fig. 1: $AY = AZ,\ CY = CX,\ BX = BZ$, being tangents from a point to the circle.

   Then

$$2AY + 2CX + 2BX = a + b + c = 2s,$$
$$\therefore AY = s - (CX + BX) = s - a,$$

   similarly the others.

   Theorem II.

$$\begin{aligned} AY_1 &= AZ_1 \\ \text{so} \qquad 2AY_1 &= AC + CY_1 + AB + BZ_1 \\ &= AC + AB + CX_1 + BX_1 = b + c + a = 2s_1 \\ \text{and} \qquad CY_1 &= AY_1 - AC = s - b. \end{aligned}$$

2. Using Theorem I and II.

$$\begin{aligned} CY &= s - c \\ CY_1 &= s - b \therefore \quad CY + CY_1 \quad = YY_1 = (s - c) + (s - b) = BX_1 + CX_1 = BC. \end{aligned}$$

   We first draw BC to be in the position of YY_1, then $YI = \delta,\ Y_1E_1 = \delta_1$, both perpendicular to YY_1. So we can draw the incircle and excircle, then CB is a common internal tangent and AB is the line of a common external tangent.