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THE BACHET EQUATION

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It is impossible to overestimate our debt to the ancient Greeks in a wide range of subjects including mathematics. Some people, however, seem to believe that the Greek contribution to mathematics was only in the area of geometry. It is interesting to note that many algebraic identities such as $(x+y)^2 = x^2 + y^2 + 2xy$ were well known to the ancient Greeks, and Euclid proves that $xy + \left(\frac{x-y}{2}\right)^2 = \left(\frac{x+y}{2}\right)^2$ in Book II, proposition 5. He does this, of course, geometrically using areas of squares and rectangles. However, by the time we come to the Greek mathematician Diophantus (c.250 AD), the geometric aspects of the mathematics he presents have almost completely disappeared, and he is thinking 'algebraically' rather than geometrically. Diophantus poses a large collection of 'number theory' problems which he then proceeds to solve, giving fairly general methods, but applied to specific examples. Diophantus did not have algebra, as we understand it, at his disposal. He used the symbol ς for our x, along with some other symbols for powers, e.g. $\triangle^Y$ (from the Greek word $\delta\upsilon\nu\alpha\mu\iota\varsigma$, dunamis, meaning 'power' whence our word 'dynamite') for x^2 and letters of the Greek alphabet instead of our number system. For example $\triangle^Y\bar{\gamma}\varsigma\bar{\delta} \Uparrow \dot{M}\bar{\epsilon}$, was his way of writing $3x^2 + 4x - 5$. Despite this clumsy and cumbersome notation he was able to do some quite sophisticated mathematics. In what follows, I use modern notation.

Diophantus' work, (or what survives of it) formed the basis for modern algebra and number theory, and was seriously studied by many mathematicians including Fermat and Euler. Attempts to generalise some of his examples led to famous and difficult problems including Fermat's Last Theorem which was only recently solved. In Book II problem 8, Diophantus posed the problem:

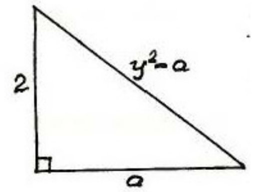
"To divide a given square number into two squares", and proceeds to divide 16 as $4^2 = \left(\frac{16}{5}\right)^2 + \left(\frac{12}{5}\right)^2$. Fermat read this and noted in the margin of his copy of the book that it was not possible to divide a cube into two cubes, a biquadrate (fourth power) into two biquadrates etc. In modern terms, he claimed that $x^n + y^n = z^n$ has no positive integer solutions for $n > 2$. As is well known, he claimed to have a proof but didn't write it down. This is the famous Last Theorem of Fermat, but it's statement was motivated by a problem in Diophantus.

A much less well known, but perhaps equally difficult problem which has not been completely solved, concerns the solutions (if any) of the Bachet equation (sometimes called Mordell's equation), $y^2 = x^3 + k$.

To be more precise, given k, an integer, when does this equation have integer solutions?

The question first arose (as far as I know) in a problem in **Diophantus**, Book VI problem 17, where he asks how:

"To find a right-angled triangle such that the area added to the hypotenuse gives a square, while the perimeter is a cube".



Diophantus' solution is very clever. He sets up the triangle with shorter sides 2 and a and hypotenuse $y^2 - a$, so the area added to the hypotenuse is y^2, i.e. a square. To satisfy the second condition he requires the perimeter, $2 + y^2$ to be a cube,

i.e. he wants $2 + y^2 = x^3$ or $y^2 = x^3 - 2$ (which is the Bachet equation with $k = -2$).

At this stage Diophantus cleverly puts $y = m + 1$ and $x = m - 1$ giving the cubic equation

$$m^3 - 3m^2 + 3m - 1 = m^2 + 2m + 3$$

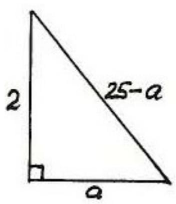
and then states that $m = 4$ without giving any reasons. As you may know, the cubic was not generally solved till about 1500 AD, so how did Diophantus get $m = 4$? Possibly, he wrote the equation as

$$m^3 + m = 4m^2 + 4$$

so

$$m(m^2 + 1) = 4(m^2 + 1)$$

and equated the factors. As he didn't know about complex numbers, he ignored the other roots $m = \pm i$.



Anyway, he thus found the solution $y = 5,\ x = 3$, and so his triangle has sides $2, a, 25-1$. By Pythagoras' Theorem he then obtains $a = \frac{621}{50}$. Notice that Diophantus allowed fractional solutions to his problems, but not negatives.

In fact $x = 3,\ y = \pm 5$ are the **only** integer solutions to the equation $y^2 = x^3 - 2$. It is not clear whether Diophantus knew this or not, and it was not proven 'till the 18th century. The proof was due to Euler, but Fermat had noted the fact (without proof) in the margin of his copy of Diophantus.

The edition of the Greek text of **Diophantus** that Fermat used was prepared by Claude-Gasper Bachet, a contemporary of Fermat, and appeared in 1621. This edition along with Fermat's marginal notes was reprinted by Fermat's son, after his death. It is from this that the Bachet equation got its name. If k is a square or a cube then it is sometimes possible to solve the equation without too much work. Fermat was able to show for example that $y^2 = x^3 - 4$ has $(x, y) = (2, \pm 2),\ (5, \pm 11)$ as its only solutions. For many values of k, e.g. $k = -5, -6, -10, -14$ etc. the equation has no integer solutions.

For many years people worked on the equation for different values of k and either found no solutions or only a finite (and generally small) number of integer solutions, but no general results or methods were found. One such person was L.J. Mordell (1888-1972) who solved the problem for many values of k and attempted to find some general properties of the solutions. He proved that for given k the equation has only finitely many solutions. Finally in 1966 Alan Baker of Cambridge University was able to give a finite procedure for determining all the solutions for a given k. This procedure was, however, very complicated and it took a little time before a practical computational method could be developed for actually implementing Bakers' ideas. They were first applied to the equation $y^2 = x^3 - 28$ to get the (complete list of) solutions $(x, y) = (4, \pm 6),\ (8, \pm 22),\ (37, \pm 225)$. The remaining unsolved problem concerning the equation is to find a simple general condition on k which determines whether or not the equation has integer solutions.