UNSW SCHOOL MATHEMATICS COMPETITION 1995
SOLUTIONS

JUNIOR DIVISION

Q.1. Let n be a positive integer. If the polynomial

$$(x+1)(x+2)(x+3)\cdots(x+n)$$

is expanded, **(a)** find the sum of all the coefficients; **(b)** find the sum of the coefficients of odd powers of x.

ANS. Let the expanded polynomial be

$$p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \ .$$

(a) The sum of all the coefficients is

$$\begin{aligned} a_0 + a_1 + a_2 + \cdots + a_n = p(1) &= 2 \times 3 \times 4 \times \cdots \times (n+1) \\ &= (n+1)! \ . \end{aligned}$$

(b) Let $x = -1$. We have

$$p(-1) = a_0 - a_1 + a_2 - a_3 + \cdots \pm a_n \ ,$$

where the final sign is $+$ if n is even and $-$ if n is odd. Thus

$$p(-1) = (a_0 + a_2 + \cdots) - (a_1 + a_3 + \cdots) \ ,$$

the sum of the even coefficients minus the sum of the odd coefficients. On the other hand,

$$p(x) = (x+1)(x+2)(x+3)\cdots(x+n)$$

and so $p(-1) = 0$. Hence the sums of even and of odd coefficients are the same, and each equals $\frac{1}{2}(n+1)!$.

Q.2. Let $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ be positive real numbers. Prove that at least one of the numbers

$$\frac{a_1}{b_1} \ , \ \frac{a_2}{b_2} \ , \ \ldots \ , \ \frac{a_n}{b_n}$$

is greater than or equal to

$$\frac{a_1 + a_2 + \cdots + a_n}{b_1 + b_2 + \cdots + b_n} \ .$$

ANS. Let $\frac{a_k}{b_k}$ be the largest of the n fractions. We have

$$\begin{aligned} &\frac{a_k}{b_k} - \frac{a_1 + a_2 + \cdots + a_n}{b_1 + b_2 + \cdots + b_n} \\ &= \frac{a_kb_1 + a_kb_2 + \cdots + a_kb_n - a_1b_k - a_2b_k - \cdots - a_nb_k}{b_k(b_1 + b_2 + \cdots + b_n)} \\ &= \frac{(a_kb_1 - a_1b_k) + (a_kb_2 - a_2b_k) + \cdots + (a_kb_n - a_nb_k)}{b_k(b_1 + b_2 + \cdots + b_n)} \qquad (*) \end{aligned}$$

Now

$$a_k b_1 - a_1 b_k = b_1 b_k \left(\frac{a_k}{b_k} - \frac{a_1}{b_1} \right) \geq 0$$

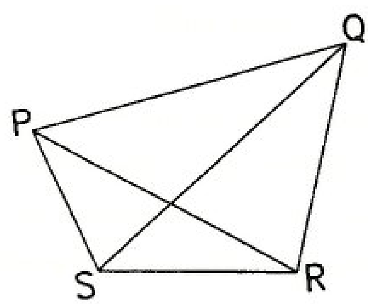
since $\frac{a_k}{b_k}$ is the largest of the n fractions; by a similar argument, every term in the numerator of $(*)$ is positive or zero. Hence

$$\frac{a_k}{b_k} \geq \frac{a_1 + a_2 + \cdots + a_n}{b_1 + b_2 + \cdots + b_n}$$

as required.

Q.3. Four points are located in a plane. For each point, the sum of the distances to the other three is calculated; and these four sums are found to be the same. Determine all possible configurations of the four points.

ANS. Consider a quadrilateral as shown.



It is given that

$$PQ + PR + PS = PQ + QR + QS = PR + QR + RS = PS + QS + RS \ .$$

Consequently

$$(PQ + PR + PS) + (PQ + QR + QS) = (PR + QR + RS) + (PS + QS + RS)$$

and simplifying yields $PQ = RS$. Similarly $PR = QS$ and $PS = QR$. Therefore all four triangles $\triangle QPS$, $\triangle PQR$, $\triangle SRQ$ and $\triangle RSP$ are congruent (three equal sides) and the four angles $\angle QPS$, $\angle PQR$, $\angle SRQ$ and $\angle RSP$ are equal. So $PQRS$ is a rectangle.

Q.4. Let n be a positive integer. Find the number of ordered triples of integers (x, y, z) for which all four of the inequalities

$$x \geq 0, \ y \geq 0, \ z \geq 0 \quad \text{and} \quad x + y + z \leq n$$

are true.

ANS. First suppose that $x = 0$. Then we have

$$y \geq 0\,, \quad z \geq 0 \quad \text{and} \quad y + z \leq n\,.$$

The solutions of these inequalities are

$$\begin{array}{l} (n,0)\,; \\ (n-1,0)\,,\ (n-1,1)\,; \\ \quad\vdots \\ (1,0)\,,\ (1,1)\,,\ (1,2)\,,\ \ldots\,,\ (1,n-1)\,; \\ (0,0)\,,\ (0,1)\,,\ (0,2)\,,\ \ldots\,,\ (0,n-1)\,,\ (0,n)\,; \end{array}$$

and the number of solutions is

$$1 + 2 + \cdots + n + (n+1) = \tfrac{1}{2}(n+1)(n+2)\,.$$

Next let $x = 1$. Then the inequalities to be solved are

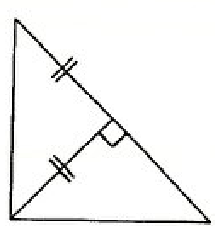
$$y \geq 0\,, \quad z \geq 0 \quad \text{and} \quad y + z \leq n - 1\,,$$

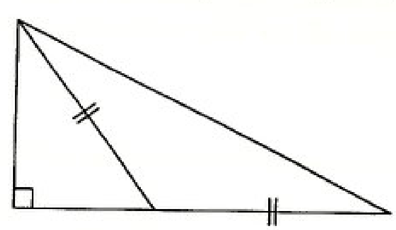
and by the same argument the number of solutions is $\frac{1}{2}n(n+1)$. Continuing in the same way, the total number of solutions is

$$\begin{aligned} &\tfrac{1}{2}(n+1)(n+2) + \tfrac{1}{2}n(n+1) + \tfrac{1}{2}(n-1)n + \cdots + \tfrac{1}{2} \times 2 \times 3 + \tfrac{1}{2} \times 1 \times 2 \\ &\quad = \tfrac{1}{6}\big[(n+1)(n+2)((n+3) - n) + n(n+1)((n+2) - (n-1)) \\ &\qquad\quad + (n-1)n((n+1) - (n-2)) + \cdots + 2 \times 3 \times (4-1) \\ &\qquad\quad + 1 \times 2 \times (3-0)\big] \\ &\quad = \tfrac{1}{6}\big[(n+1)(n+2)(n+3) - n(n+1)(n+2) \\ &\qquad\quad + n(n+1)(n+2) - (n-1)n(n+1) \\ &\qquad\quad + (n-1)n(n+1) - (n-2)(n-1)n \\ &\qquad\quad + \cdots + 2 \times 3 \times 4 - 1 \times 2 \times 3 \\ &\qquad\quad + 1 \times 2 \times 3 - 0 \times 1 \times 2\big] \\ &\quad = \tfrac{1}{6}(n+1)(n+2)(n+3)\,. \end{aligned}$$

Q.5. Prove that if $n \geq 4$ then any triangle can be dissected into n isosceles triangles.

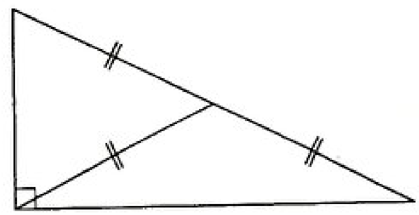
ANS. First observe that a right-angled triangle can be dissected into an isosceles triangle and a right-angled triangle. The following diagrams show how this is done, firstly in the case of a right-angled isosceles triangle (where in fact each subtriangle is both right-angled and isosceles), and secondly in the case of any other right-angled triangle.





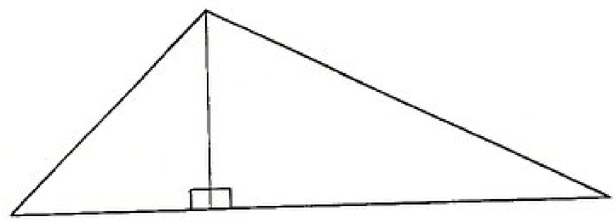
By repeating this operation on the smaller right-angled triangle (or by leaving the original triangle untouched) we see that a right-angled triangle can be dissected into a right-angled triangle and any number of isosceles triangles (including zero).

Secondly, a right-angled triangle can be divided into two isosceles triangles as shown.



Combining this with the previous dissection we get a dissection of any right-angled triangle into two or more isosceles triangles.

Finally, any triangle can be dissected into two right-angled triangles;



if $n \geq 4$ then these two triangles can be divided into 2 and $n-2$ isosceles triangles, giving a dissection of the original triangle into n isosceles triangles.

Q. 6. A collection of 1995 numbers consists of one zero and 1994 ones.

(a) It is permitted to choose any two numbers from the collection and replace each of them by the average of the two. Is it possible by repeating this operation to obtain a collection in which all 1995 numbers are the same?

(b) It is permitted to choose any two *or more* of the numbers (but not the whole collection) and replace each of them by the average of the chosen numbers. Is it now possible to make all the numbers equal?

ANS. (a) No, it is not possible. Note that since at each stage we are replacing each of two numbers by their average, the sum of all 1995 numbers never changes. Thus if the required result were possible, all the numbers would have to end up as $\frac{1994}{1995}$. However, if we write all the original numbers as fractions then every denominator is a power of 2:

$$0 = \frac{0}{2^0}, \quad 1 = \frac{1}{2^0};$$

and if we average two fractions with powers of 2 in the denominator we obtain

$$\frac{1}{2}\left(\frac{a}{2^m} + \frac{b}{2^n}\right) = \frac{2^n a + 2^m b}{2^{m+n+1}},$$

which still has a power of 2 for its denominator. Thus it is impossible to reach a fraction such as $\frac{1994}{1995}$ in which the denominator is not a power of 2.

(b) Yes, this is possible. First replace $0, 1, 1, 1$ and 1 by their average. Now five of the numbers are $\frac{4}{5}$ and the other 1990 are ones. We can split this collection up into five groups, each consisting of a $\frac{4}{5}$ and 398 ones; and the average of each group is

$$\frac{1}{399}\left(398+\frac{4}{5}\right)=\frac{1994}{1995}\,.$$

Thus all numbers in the collection are now the same.

* * * * * * * *

Continued from p.4

which is Heron's formula.

Exercises:

1. Prove Theorems I and II from the article about Heron's formula.
2. Given the lengths of ρ, ρ_1 and BC, describe how to construct $\triangle ABC$, using only a compass and a straightedge.

SOLUTIONS TO SENIOR DIVISION QUESTIONS

Q.1. Let $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ be positive real numbers. Prove if the numbers

$$\frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_n}{b_n}$$

are not all equal, then at least one of them is greater than the fraction

$$\frac{a_1 + a_2 + \cdots + a_n}{b_1 + b_2 + \cdots + b_n}$$

and at least one is less than this fraction.

ANS. 1 Suppose a, b, c, d are positive real numbers and $\frac{a}{b} < \frac{c}{d}$ then we shall show $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$. Suppose $ad < bc$, then $ad+ab < bc+ab$, $a(b+d) < b(a+c)$, $\frac{a}{b} < \frac{a+c}{b+d}$. Similarly $ad < bc$, $ad + cd < bc + cd$, $d(a+c) < c(b+d)$, $\frac{a+c}{b+d} < \frac{c}{d}$.

Assume $\frac{a_i}{b_i} \le \frac{a_{i+1}}{b_{i+1}}$, $i = 1, \cdots, n-1$, by relabelling. Now since the n quotients are not equal there is a pair with $\frac{a_i}{b_i} < \frac{a_i + a_{i+1}}{b_i + b_{i+1}} < \frac{a_{i+1}}{b_{i+1}}$.

Hence by $i-1$ steps

$$\frac{a_{i-1}}{b_{i-1}} < \frac{a_{i-1} + a_i + a_{i+1}}{b_{i-1} + b_i + b_{i+1}} < \frac{a_i + a_{i+1}}{b_i + b_{i+1}}$$

$$\frac{a_1}{b_1} < \frac{a_1 + \cdots + a_{i+1}}{b_1 + \cdots + b_{i+1}} < \cdots \le \frac{a_{i+2}}{b_{i+2}}$$

Next by $n-i$ steps

$$\frac{a_1}{b_1} < \frac{a_1 + \cdots + a_{i+2}}{b_1 + \cdots + b_{i+2}} < \frac{a_{i+2}}{b_{i+2}} \le \frac{a_{i+3}}{b_{i+3}}$$

$$\frac{a_1}{b_1} < \frac{a_1 + a_2 + \cdots + a_n}{b_1 + b_2 + \cdots + b_n} < \frac{a_n}{b_n}.$$

ALT. ANS.

Suppose, without loss of generality, that

$$k = \frac{a_1}{b_1} \le \frac{a_2}{b_2} \le \cdots \le \frac{a_n}{b_n}.$$

Then $a_1 = kb_1$ then $a_2 \ge kb_2$, $a_3 \ge kb_3 \ge \cdots a_n > kb_n$ since at least one quotient is $> k$.

Hence

$$\frac{a_1 + \cdots + a_n}{b_1 + \cdots + b_n} > \frac{kb_1 + kb_2 + \cdots + kb_n}{b_1 + b_2 + \cdots + b_n} = k = \frac{a_1}{b_1}.$$

Similarly

$$\frac{a_n}{b_n} > \frac{a_1 + \cdots + a_n}{b_1 + \cdots + b_n}.$$

Q.2. A collection of 1995 numbers consists of one zero and 1994 ones.

(a) It is permitted to choose any two numbers from the collection and replace each of them by the average of the two. Is it possible by repeating this operation to obtain a collection in which all 1995 numbers are the same?

(b) It is permitted to choose any two *or more* of the numbers (but not the whole collection) and replace each of them by the average of the chosen numbers. Is it now possible to make all the numbers equal?

ANS. 2a

The collection initially consists of one 0 and 1994 1's. The first step either leaves the set unchanged or yields two 1/2's and 1993 1's. The average of the set is 1994/1995 and if the numbers are all equal this must be their value. However $2^{10} < 1995 < 2^{11}$ and the process only generates numbers of the form $\frac{n}{2^a}$ since $\frac{1}{2}(\frac{n}{2^a} + \frac{m}{2^b}) = \frac{s}{2^c}$ some s, c. Hence the process does not work.

ANS. 2b

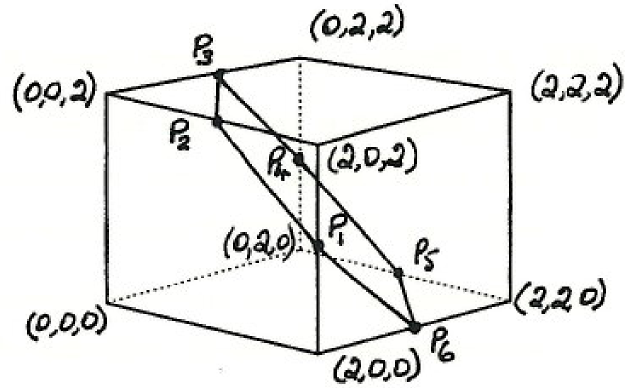
$1995 = 5.399$. Proceed as follows:

Take the zero and four 1's, average them producing five 4/5's. Next group each 4/5 with 398 1's and average them yielding 399 1994/1995's. Hence we now have all the numbers equal. Clearly the method works for any n provided n (in this case 1995) is not prime.

Q.3. Show that it is possible for a cube and a plane to intersect in a regular hexagon, but impossible for a cube and a plane to intersect in a regular pentagon.

ANS. 3

Consider the cube of side length 2 with vertices $(0,0,0)$, $(0,0,2)$, $(0,2,0)$, $(0,2,2)$, $(2,0,0)$, $(2,0,2)$, $(2,2,0)$ and $(2,2,2)$.



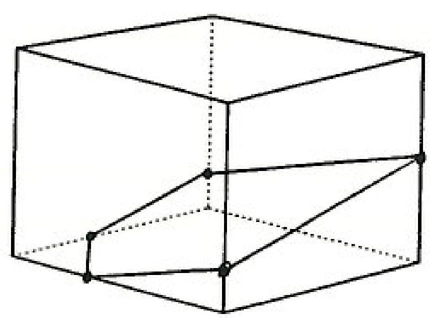
Every plane in $\mathbf{R}^3$ is of the form $ax + by + cz = d$. (This is obvious if $c = 0$ when one obtains a plane parallel to the z-axis meeting the x, y-plane in $ax + by = d$!).

Consider the plane $x + y + z = 3$! This meets the edges of the cube in six points $P_1 = (2,0,1)$, $P_2 = (1,0,2)$, $P_3 = (0,1,2)$, $P_4 = (0,2,1)$, $P_5 = (1,2,0)$ and $P_6 = (2,1,0)$. It also passes through $C = (1,1,1)$ the centre of the cube. Now if d is the distance between $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$ then $d^2 = (a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2$ hence

$$\begin{aligned}(P_1P_2)^2 &= P_2P_3^2 = P_3P_4^2 = P_4P_5^2 = P_5P_6^2 = P_6P_1^2 \\ &= CP_1^2 = CP_2^2 = CP_3^2 = CP_4^2 = CP_5^2 = CP_6^2 = 2.\end{aligned}$$

Thus $P_1P_2 \cdots P_6$ is a regular hexagon, with edge length $\sqrt{2}$.

A plane can meet a cube in a pentagon, for example,



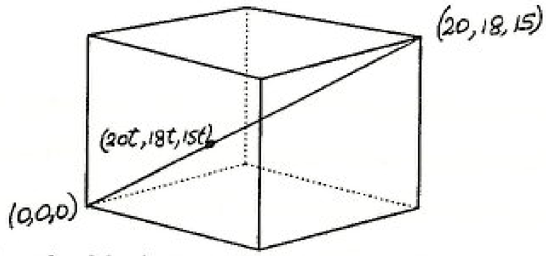
However, if it does four of the sides form parallel pairs. Thus the pentagon is not regular since in a regular pentagon all angles are $108° = 3\pi/5$. Incidentally a plane can also meet a cube in an equilateral triangle (near a vertex) or a square.

Q.4 **(a)** Unit cubes are arranged into an $20 \times 18 \times 15$ block. A straight line is drawn from one corner of the block to the diagonally opposite corner. How many unit cubes does the line pass through?

(b) Repeat the question if the block has dimensions $a \times b \times c$.

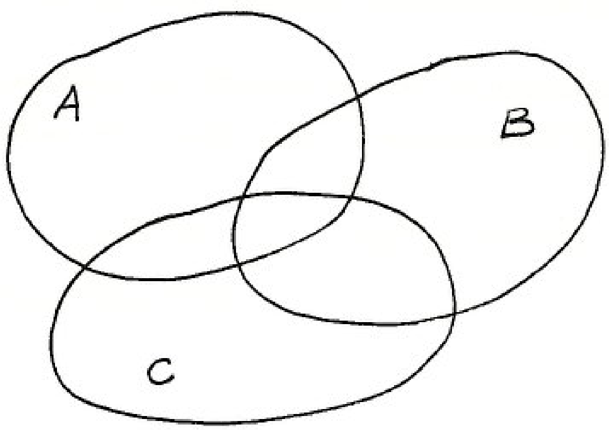
ANS. 4a

By similar triangles it is easy to see that the typical point on the long diagonal of the cube has coordinates $(20t,\ 18t,\ 15t)$ with $0 \le t \le 1$.



The planes dividing the block into cubes are $x = i,\ 0 \le i \le 20;\ y = j,\ 0 \le j \le 18$ and $z = k,\ 0 \le k \le 15$.

Hence, the line moves from one block to another when $t = \frac{i}{20}$ or $\frac{j}{18}$ or $\frac{k}{15}$. We shall count the number of cubes entered, that is, the number of values of t of the given form, $0 \le t < 1$. Let A be the number of multiples m of $\frac{1}{20}$ with $0 \le m < 1$, B multiples of $\frac{1}{18}$, C multiples of $\frac{1}{15}$.



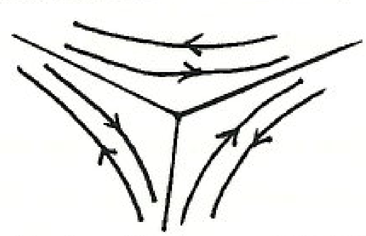
$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Now $|A| = 20$, $|B| = 18$, $|C| = 15$. If m is a multiple of $\frac{1}{20}$ and $\frac{1}{18}$, then $m = 0$ or $1/2$ so $|A \cap B| = 2$. Similarly $|A \cap C| = 5$, $|B \cap C| = 3$ and $|A \cap B \cap C| = 1$. So $|A \cup B \cup C| = 20 + 18 + 15 - 2 - 3 - 5 + 1 = 44$. So the line passes through 44 cubes.

Q.5. On an island there are a number of towns, and a number of roads linking the towns. Each town is the junction of exactly three roads. A traveller sets out along a road from one town, and at the next town takes the left hand road of the two available. At the following town the right hand road is taken, and so on, with left and right turns alternating. Prove that at some stage the traveller must return to the first town.

ANS. 5

The number of towns is finite and there are only six ways of entering and leaving any given town. Hence the traveller must eventually pass through some town T twice entering and leaving in the same direction. From this point on the traveller loops for ever.



Suppose that after two loops the traveller turns around and retraces his steps – turning left where he turned right and vice versa. This reverses the journey staying on the loop hence the initial town is on the loop at least once and up to six times!

Q.6. If x is a real number, $\lceil x \rceil$ denotes x rounded up to the next integer. (If x is itself an integer then $\lceil x \rceil$ is the same as x.) Prove that if n is a positive integer then

$$\left(\left\lceil \sqrt[1995]{1} \right\rceil + \left\lceil \sqrt[1995]{2} \right\rceil + \left\lceil \sqrt[1995]{3} \right\rceil + \cdots + \left\lceil \sqrt[1995]{n^{1995}} \right\rceil\right) + \left(1^{1995} + 2^{1995} + 3^{1995} + \cdots + n^{1995}\right)$$

is equal to

$$n^{1996} + n^{1995} .$$

ANS. 6

$$\left\lceil \sqrt[1995]{1} \right\rceil = 1$$

$$\left\lceil \sqrt[1995]{2} \right\rceil = 2$$

Also

$$\lceil x \rceil = 2 \quad \text{if} \quad 2 \le x \le 2^{1995}$$

$$\lceil x \rceil = 3 \quad \text{if} \quad 2^{1995} + 1 \le x \le 3^{1995}$$

$$\lceil x \rceil = n \quad \text{if} \quad (n-1)^{1995} + 1 \le x \le n^{1995}$$

Hence the sum

$$= 1 + (2^{1995} - 1)2 + (3^{1995} - 2^{1995})3 + \cdots + (n^{1995} - (n-1)^{1995})n$$
$$+ 1^{1995} + 2^{1995} + \cdots + (n-1)^{1995} + n^{1995}$$
$$= 1 - 2 + 1 + 2^{1995}(2 - 3 + 1) + 3^{1995}(3 - 4 + 1) + \cdots$$
$$+ (n-1)^{1995}((n-1) - n + 1) + n^{1995}(n+1)$$
$$= n^{1996} + n^{1995}.$$

This tomb holds Diophantus. Ah, how great a marvel! The tomb tells scientifically the measure of his life. God granted him to be a boy for the sixth part of his life, and adding a twelfth part to this, He clothed his cheeks with down; He lit him the light of wedlock after a seventh part, and five years after his marriage He granted him a son. Alas! lateborn wretched child; after attaining the measure of half his father's life, chill Fate took him. After consoling his grief by this science of numbers for four years he ended his life.