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SOLUTIONS TO PROBLEMS 941-948

Q.941 On my desk calendar two numbers are given: the number of days in the year up to today, and the number remaining. (For example, on New Year's Day the numbers were 1 and 364.) What date could it be if the two numbers have the same digits, possibly in a different order? (No number may start with a zero.) Will the answer to this question be the same next year?

ANS. Let the numbers be abc and def, where a, b, c and d, e, f are the same digits (except for order). Writing out the sum of the two numbers

$$\begin{array}{cccc} & a & b & c \\ + & d & e & f \\ \hline & 3 & 6 & 5 \end{array}$$

and remembering that 1 may need to be carried from one column to the next, we see that $a+d$, $b+e$ and $c+f$ are respectively

$$3, 6, 5 \quad \text{or} \quad 2, 16, 5 \quad \text{or} \quad 3, 5, 15 \quad \text{or} \quad 2, 15, 15.$$

Since every digit occurs twice, the sum $a+b+c+d+e+f$ is even and we can reject the second and third cases. To obtain the 3 in the first case two of the digits must be 1 and 2; the sum of the digits, taken twice each, is $3+6+5=14$; so the third digit is 4. This gives the answer

$$124 + 241 = 365.$$

In the fourth case $a = d = 1$ and the other two digits add up to 15, so we have

$$169 + 196 = 365 \quad \text{or} \quad 187 = 365.$$

So the possible day numbers are

$$124, 169, 178, 187, 196, 241$$

giving dates of May 4, June 18, June 27, July 6, July 15 and August 29.

Next year the same ideas, together with the fact that the year has 366 days, will give the unique answer of July 1.

Q.942 (i) Find positive integers x, y such that

$$x^2 - 2y^2 = -1.$$

(ii) Show that if (x, y) is a solution of the above equation then $(3x+4y,\ 2x+3y)$ is also a solution.

(iii) Prove that there are infinitely many non-negative integers n such that $n^2 + (n+1)^2$ is a square.

ANS. (i) By trial and error, $x = y = 1$ will do.

(ii) If $x^2 - 2y^2 = -1$ then

$$(3x+4y)^2 - 2(2x+3y)^2 = (9x^2 + 24xy + 16y^2) - (8x^2 + 24xy = 18y^2)$$
$$= x^2 - 2y^2$$
$$- 1.$$

(iii) Consider the equation

$$n^2 + (n+1)^2 = m^2. \qquad (*)$$

Multiplying both sides by 2 and rearranging, we have

$$(2n+1)^2 - 2m^2 = -1.$$

Now let $x = 2n+1,\ y = m$. From (i), the equation $x^2 - 2y^2 = -1$ has a solution; and from (ii), any solution leads to a larger solution. Thus the equation has infinitely many solutions. Also it is clear that x is odd, so n is an integer. Hence $(*)$ has infinitely many non-negative integer solutions.

Comment. From (i) and (ii) we can find the first few solutions

$$(x, y) = (1,1),\ (7,5),\ (41,29),\ (239,169), \cdots$$

which give

$$(n, m) = (0,1),\ (3,5),\ (20,29),\ (119,169), \cdots,$$

and we may check that

$$0^2+1^2=1^2,\ 3^2+4^2=5^2,\ 20^2+21^2=29^2,\ 119^2+120^2=169^2,\cdots$$

Q.943 Prove that if $a_2, a_2, \cdots, a_n$ and $x_1, x_2, \cdots, x_n$ are positive numbers then

$$(a_1x_1+a_2x_2+\cdots+a_nx_n)(\frac{a_1}{x_1}+\frac{a_2}{x_2}+\cdots+\frac{a_n}{x_n}) \geq (a_1+a_2+\cdots+a_n)^2.$$

Under what conditions does equality hold?

ANS. First note that if $x>0$ then

$$x+\frac{1}{x}=2+\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2 \geq 2,$$

with equality when $\sqrt{x}-\frac{1}{\sqrt{x}}=0$, that is, $x=1$. Now expand the given product. We obtain n terms

$$a_1^2,\ a_2^2,\cdots,a_n^2;$$

and whenever $i<j$ there are also two terms

$$a_ix_k\frac{a_j}{x_j}+a_jx_j\frac{a_i}{x_i}=a_ia_j\left(\frac{x_i}{x_j}+\frac{x_j}{x_i}\right)\geq 2a_ia_j,$$

using the above result with $x=\frac{x_i}{x_j}$. Hence the LHS is at least

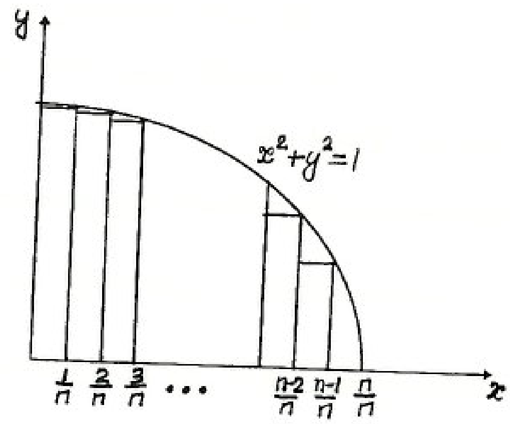
$$a_1^2+a_2^2+\cdots+a_n^2+2a_1a_2+2a_1a_3+\cdots+2a_{n-1}a_n$$

which is the expansion of $(a_1+a_2+\cdots+a_n)^2$. The two sides are equal if and only if $\frac{x_i}{x_j}=1$ for every i,j; that is, if $x_1, x_2, \cdots, x_n$ are all equal.

Q.944 If n is large, find a simple approximate formula for

$$\sqrt{1-\frac{1}{n^2}}+\sqrt{1-\frac{4}{n^2}}+\sqrt{1-\frac{9}{n^2}}+\cdots+\sqrt{1-\frac{(n-1)^2}{n^2}}+\sqrt{1-\frac{n^2}{n^2}}$$

ANS. Consider the following diagram.



The kth rectangle from the left has height equal to the y-coordinate on the circle when $x = \frac{k}{n}$, that is,

$$y = \sqrt{1 - \frac{k^2}{n^2}}$$

and therefore has area $\frac{1}{n}\sqrt{1 - \frac{k^2}{n^2}}$. (The nth rectangle has height 0 and is therefore invisible.)

It is clear from the diagram that adding the areas of all these rectangles will give approximately the area of the quarter circle, and that the approximation will become better and better as n becomes larger. So if n is large we have

$$\sqrt{1 - \frac{1}{n^2}} + \sqrt{1 - \frac{4}{n^2}} + \cdots + \sqrt{1 - \frac{(n-1)^2}{n^2}} + \sqrt{1 - \frac{n^2}{n^2}} = \frac{n\pi}{4}$$

approximately.

Q.945 A regular polygon with n sides is inscribed in a circle. If A, B, C and D are four successive vertices of the polygon then the length of AD equals the side of the polygon plus the radius of the circle. Find all possible values of n.

ANS. Let the radius of the circle be r. The angle subtended at the centre by one side is 2θ, where $\theta = \frac{\pi}{n}$. The side length will be $2r\sin\theta$, and the length of a chord compassing three sides (AD in the question) will be $2r\sin 3\theta$. Using the formula

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$$

we have

$$2r\sin 3\theta = 2r\sin\theta + r$$

$$\Rightarrow \quad 6\sin\theta - 8\sin^3\theta = 2\sin\theta + 1$$

$$\Rightarrow \quad \sin^3\theta - \tfrac{1}{2}\sin\theta + \tfrac{1}{8} = 0.$$

This is a cubic in $\sin\theta$; a little trial and error gives $\sin\theta = \frac{1}{2}$ as one solution. So

$$(\sin\theta - \tfrac{1}{2})(\sin^2\theta + \tfrac{1}{2}\sin\theta - \tfrac{1}{4}) = 0$$

and apart from the solution $\sin\theta = \frac{1}{2}$ we have also the solutions of the quadratic

$$\sin\theta = \frac{-\frac{1}{2} \pm \sqrt{(\frac{1}{2})^2 + 1}}{2} = \frac{-1 \pm \sqrt{5}}{4}.$$

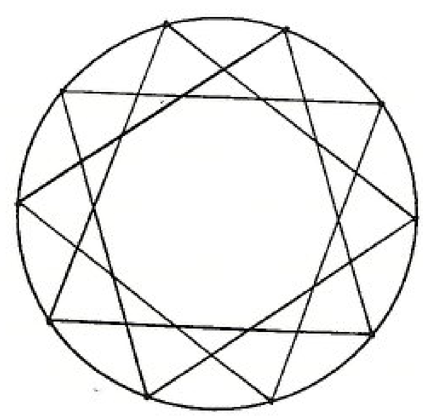
It was shown in an earlier problem (see p.32 of **Parabola** 29(1)) that

$$\sin\frac{\pi}{10} = \frac{-1+\sqrt{5}}{4}, \; \sin\frac{3\pi}{10} = \frac{1+\sqrt{5}}{4}.$$

Thus the solutions are

$$\theta = \frac{\pi}{6}, \frac{\pi}{10}, -\frac{3\pi}{10}$$

and the polygon has $6, 10$ or $-\frac{10}{3}$ sides. The third of these answers, however, makes no sense geometrically. Or does it? You might like to consider the following diagram.



Q.946 (i) Show that if n is an integer, $n \geq 0$, then $2^{4n+2} + 1$ is divisible by 5.

(ii) Factorise $x^{4n} + 4$ into a product of two polynomials.

(iii) Show that if $n \geq 2$ then $\dfrac{2^{4n+2}+1}{5}$ is composite.

ANS. (i) We have

$$2^{4n+2} + 1 = 4 \times 16^n + 1 = 4(15+1)^n + 1.$$

By the Binomial theorem,

$$\begin{aligned}(15+1)^n &= 15^n + \binom{n}{1} 15^{n-1} + \cdots + \binom{n}{n-1} 15 + 1 \\ &= (\text{a multiple of } 5) + 1;\end{aligned}$$

hence

$$\begin{aligned}2^{4n+2} + 1 &= 4 \times ((\text{a multiple of } 5) + 1) + 1 \\ &= (\text{a multiple of } 5) + 4 + 1\end{aligned}$$

which is divisible by 5.

(ii) We can write the polynomial as a difference of two squares:

$$\begin{aligned}x^{4n} + 4 &= (x^{2n} + 2)^2 - 4x^{2n} \\ &= (n^{2n} + 2x^n + 2)(x^{2n} - 2x^{+}2).\end{aligned}$$

(iii) Substituting $x = 2$ and replacing n by $n+1$ in the previous part,

$$2^{4n+4} + 4 = (2^{2n+2} + 2^{n+2} + 2)(2^{2n+2} - 2^{n+2} + 2)$$

and so

$$2^{4n+2} + 1 = (2^{2n+1} + 2^{n+1} + 1)(2^{2n+1} - 2^{n+1} + 1).$$

Now we know from (i) that 5 is a factor of the left hand side; so if the quotient is not composite then the factorisation on the right hand side must be

$$5 \times 1 \quad \text{or} \quad (\text{a prime}) \times 5.$$

Hence the smaller factor is less than or equal to 5, which is impossible since

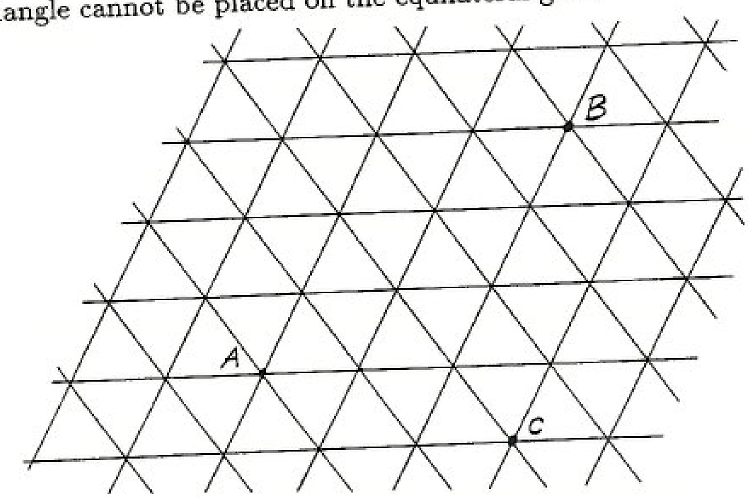
$$\begin{aligned}2^{2n+1} - 2^{n+1} + 1 &= (2^n - 1)2^{n+1} + 1 \\ &\geq 3 \times 8 + 1\end{aligned}$$

for $n \geq 2$.

Comment. If $n = 0$ or 1 then by direct calculation $\dfrac{2^{4n+2}+1}{5} = 1$ or 13.

Q.947 Three series of equidistant parallel lines are drawn in a plane, each series forming an angle of 60° with the other two; the plane is thus covered with a network of equilateral triangles. Is it possible to find four of the intersection points of these lines which form the vertices of a square?

ANS. It is impossible. This follows immediately when we show that a right-angled isosceles triangle cannot be placed on the equilateral grid.



Suppose, on the contrary, that the points A, B, C in the diagram form an isosceles triangle with a right angle at C. Referring to the x and y directions as shown, let the position of B relative to A be p units in the x direction and q in the y direction; and the position of C relative to A, r units in the x direction and the s in the y direction. (For the points shown, $p = 2, q = 3, r = 3, s = -1$.) By the cosine rule we can calculate the lengths of AB and AC :

$$\begin{aligned} AB^2 &= p^2 + a^2 - 2pq\cos 120^\circ \\ &= p^2 + a^2 + pq \end{aligned}$$

and similarly

$$AC^2 = r^2 + s^2 + rs.$$

But sinceABC is a right angled isosceles triangle, $AB^2 = 2 \times AC^2$, that is,

$$p^2 + pq + q^2 = 2(r^2 + rs + s^2). \qquad (*)$$

This shows that p^2+pq+q^2 is even. Therefore p and q must both be even (if one is even and the other odd then p^2+pq+q^2 is even + even + odd, which is odd; while if both are odd then p^2+pq+q^2 is odd + odd + odd, which again is odd).

Write $p=2p_1, q=2q_1$. Then (after simplifying)

$$r^2+rs+s^2=2(p_1^2+p_1q_1+q_1^2)$$

and the same argument shows that $r=2r_1,\ s=2s_1$. Hence

$$p_1^2+p_1q_1+q_1^2=2(r_1^2+r_1s_1+s_1^2)$$

and $p_1=2p_2,\ q_1=2q_2$. Since this argument can be repeated indefinitely, we can show that p is divisible by larger and larger numbers. But this is impossible for non-zero itegers. Therefore (*) has no integer solutions except for $p=q=r=s=0$, and it is impossible to locate a right-angled triangle (or a square) on the equilateral triangluar grid.

Q.948 An infinite sequence of real numbers $a_1, a_2, a_3, \cdots$ is defined by choosing some value of a_1 and specifying

$$a_{n+1}=\frac{a_n+c}{1-ca_n}$$

for $n\geq 1$, where c is constant. Prove that for every integer $k>2$, a constant c can be found such that the sequence is periodic and has period k. (A sequence is called periodic if at some stage it repeats itself from the beginning; the period of the sequence is the smallest possible number of steps before the repetition begins. For example the sequence $5,7,1,2,5,7,1,2,5,7,1,2,\cdots$ has period 4.)

ANS. If $k>2$, choose $c=\tan\frac{\pi}{k}$. We shall show that the sequence has period k. For each n, choose θ_n such that $a_n=\tan\theta_n$. Then the recurrence formula becomes

$$\tan\theta_{n+1}=\frac{\tan\theta_n+\tan\frac{\pi}{k}}{1-\tan\theta_n\tan\frac{\pi}{k}},$$

that is,

$$\tan\theta_{n+1}=\tan(\theta_n+\frac{\pi}{k})$$

by the "tangent of a sum" formula. Hence θ_{n+1} equals $\theta_n + \frac{\pi}{k}$ plus a multiple of π, say

$$\theta_{n+1} = \theta_n + \frac{\pi}{k} + m_n\pi$$

where m_n is an iteger. Therefore, for any $n \geq 1$ and $j \geq 1$ we have

$$\begin{aligned}
\theta_{n+j} &= \theta_{n+j-1} + \frac{\pi}{k} + m_{n+j-1}\pi \\
&= \theta_{n+j-2} + \frac{2\pi}{k} + (m_{n+j-2})\pi \\
&= \cdots \\
&= \theta_n + \frac{j\pi}{k} + (m_{n+j-1} + m_{n+j-2} + \cdots + m_n)\pi
\end{aligned}$$

and so

$$a_{n+j} = \tan\theta_{n+j} = \tan(\theta_n + \frac{j\pi}{k}).$$

Therefore

$$a_{n+k} = \tan(\theta_n + \pi) = \tan\theta_n = a_n$$

aso the sequence repeats after k steps; while if $0 < j < k$ then

$$a_{n+j} = \tan(\theta_n + \frac{j\pi}{k}) \neq \tan\theta_n,$$

so the sequence does not repeat after fewer than k steps. Thus the sequence had period k steps. Thus the sequence had period k.

Comment: it is impossible for a sequence satisfying the given recurrence to have period 2. (Exercise: prove it!)