BRAHMAGUPTA'S FORMULA

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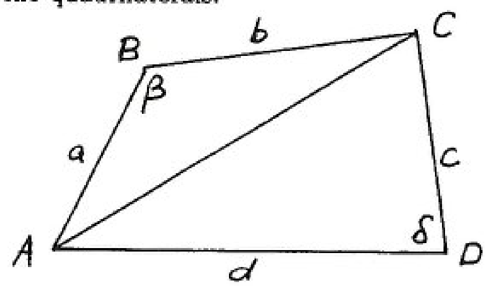
In the last two issues of **Parabola** two different proofs –one trigonometric and one geometric –were given of Heron's formula for the area of a triangle and reference was made to Brahmagupta's formula for the area of a quadrilateral. Brahmagupta lived in Central India during the 7^{th} century AD and in his best known work **Brahmasphuta Siddhanta** he published his famous formula

$$A = \sqrt{(s-a)(s-b)(s-c)(s-d)}$$

for the area of a quadrilateral where a, b, c, d are the sides and s is the semiperimeter. As we will see this is true only for cyclic quadrilaterals.



Let the quadrilateral be defined as above. Now

$$|ABC| = \frac{1}{2}ab\sin\beta,\ |ADC| = \frac{1}{2}cd\sin\delta$$

and let

$$|ABCD| \equiv A = \frac{1}{2}(ab\sin\beta + cd\sin\delta)$$

so that

$$2A = ab\sin\beta + cd\sin\delta. \qquad (1)$$

Now

$$AC^2 = a^2 + b^2 - 2ab\cos\beta \qquad = c^2 + d^2 - 2cd\cos\delta.$$

Therefore

$$a^2 + b^2 - c^2 - d^2 = 2(ab\cos\beta - cd\cos\delta). \qquad (2)$$

Hence, by (1) and (2)

$$(a^2 + b^2 - c^2 - d^2)^2 + (4A)^2$$
$$= 4(ab\cos\beta - cd\cos\delta)^2 + 4(ab\sin\beta + cd\sin\delta)^2$$
$$= 4[a^2b^2\cos^2\beta - 2abcd\cos\beta\cos\delta + c^2d^2\cos^2\delta + a^2b^2\sin^2\beta$$

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$$
\begin{aligned}
&+2abcd\sin\beta\sin\delta + c^2d^2\sin^2\delta] \\
&= 4[a^2b^2 - 2abcd(\cos\beta\cos\delta - \sin\beta\sin\delta) + c^2d^2] \\
&= 4[a^2b^2 + c^2d^2 - 2abcd\cos(\beta+\delta)] \\
&= 4[a^2b^2 + c^2d^2 - 2abcd(2cos^2\psi - 1)] \qquad \text{where } 2\psi = \beta + \delta \\
&= 4[a^2b^2 + c^2d^2 - 4abcd\cos^2\psi + 2abcd] \\
&= 4[a^2b^2 + 2abcd + c^2d^2 - 4abcd\cos^2\psi] \\
&= 4[(ab+cd)^2 - 4abcd\cos^2\psi].
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
(4A)^2 \quad = \quad & 4(a\,b + c\,d)^2 - (a^2 + b^2 - c^2 - d^2)^2 - 16a\,b\,c\,d\cos^2\psi \\
&= [2(ab+cd) - (a^2+b^2-c^2-d^2)][2(ab+cd) + (a^2+b^2-c^2-d^2)] - 16abcd\cos^2\psi \\
&= [c^2 + 2cd + d^2 - (a^2 - 2ab + b^2)][a^2 + 2ab + b^2 - (c^2 - 2cd + d^2)] - 16abcd\cos^2\psi \\
&= [(c+d)^2 - (a-b)^2][(a+b)^2 - (c-d)^2] - 16abcd\cos^2\psi \\
&= [(c+d-a+b)(c+d+a-b)][(a+b-c+d)(a+b+c-d)] - 16abcd\cos^2\psi \\
&= [(a+b+c+d-2a)(a+b+c+d-2b)][(a+b+c+d-2c)(a+b+c+d-2d)] \\
&-16abcd\cos^2\psi \\
&= [(2s-2a)(2s-2b)(2s-2c)(2s-2d)] - 16abcd\cos^2\psi \quad \text{where } s = \frac{a+b+c+d}{2} \\
&= 2^4(s-a)(s-b)(s-c)(s-d) - 16abcd\cos^2\psi.
\end{aligned}
$$

Thus, $A^2 \quad = (s-a)(s-b)(s-c)(s-d) - abcd\cos^2\psi$

We have therefore proved that the area of a quadrilateral with sides a, b, c and d is given by

$$A = \sqrt{(s-a)(s-b)(s-c)(s-d) - abcd\cos^2\psi}$$

where s is the semiperimeter and ψ is the average value of a pair of opposite angles.

When the quadrilateral is cyclic $\cos\psi = 0$ and we obtain the original formula of Brahmagupta. For such quadrilaterals it is easy to obtain a formula for the lengths of the diagonals. (This is another favourite of Brahmagupta.)

If the quadrilateral is cyclic $\cos\beta = -\cos\delta$ and equation (2) can be rewritten as:

$$a^2 + b^2 - c^2 - d^2 = 2(ab+cd)\cos\beta.$$

Therefore,

$$AC^2 \quad = \quad a^2 + b^2 - 2ab\cos\beta$$

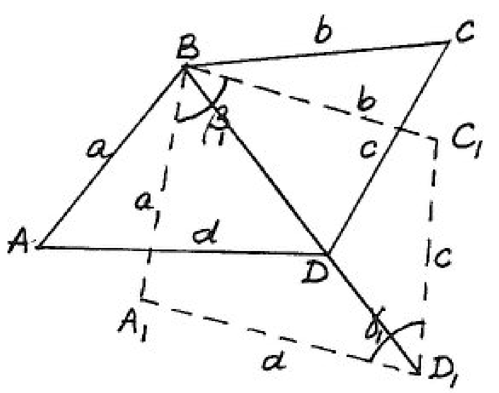
$$= a^2 + b^2 - 2ab\frac{(a^2 + b^2 - c^2 - d^2)}{2(ab + cd)}$$

$$AC^2 = \frac{(ac + bd)(ad + bc)}{(ab + cd)}$$

Notice that a corollary to Brahmagupta's formula is the following result:

A quadrilateral inscribed in a circle has an area greater than any other quadrilateral with the same sides.

It is not so clear that a quadrilateral with given sides a, b, c, d can be inscribed in a circle but here is an argument that this is the case.



Suppose $\beta + \delta > 180^o$ If we choose on BD produced an arbitrary point D_1 (such that $BD_1 \leq a + d, b + c$) then we can construct a quadrilateral A, B, C, D with sides a, b, c, d so that $\alpha_1 > \alpha$ and $\gamma_1 > \gamma$. But as D_1 varies so that $BD_1 \longrightarrow \min(a + d, b + c)$ it follows that $\beta_1 + \delta_1 \longrightarrow 0$, and in particular there is a choice of D_1 such that $\beta_1 + \delta_1 < 180^o$. But $\beta_1 + \delta_1$ is a "continuous" function of the variable $x = BD_1$. Therefore there is a choice of D_1 for which $\beta_1 + \delta_1 = 180^o$ and the resulting quadrilateral A, B, C, D, is cyclic.

Therefore we have proved:

Among the quadrilaterals having sides of the same lengths there exists a quadrilateral with maximum area. That quadrilateral is cyclic.

This theorem is a particular case of a theorem of Gabriel Cramer (1704-1752).

Among the polygons with given sides $a_1, a_2, \cdots, a_n$ (where $n \geq 3$) the polygon inscribed in a circle has the greatest area.