Parabola Volume 31, Issue 3 (1995)

DECIMAL - FORWARDS AND BACKWARDS

N.J. Wildberger*

A fraction is a number of the form $\frac{a}{b}$, where a, b are integers and $b \neq 0$. We learn to add, substract, multiply and divide fractions on our mothers knee. The set of all fractions, or rational numbers, is denoted $\mathbb{Q}$ and is closed under these operations as long as we remember that dividing by 0 is not allowed. (By "closed" under addition we just mean that if we add two rationals we get another rational.) At some later point we learn, perhaps to our consternation, that there are other numbers which do not belong to $\mathbb{Q}$, and some of them are even useful-such as $\sqrt{2}$, or π. So we must enlarge our notion of 'number' so that we will be able to incorporate and work with these irrational numbers as well.

The usual way of doing this is to consider infinite decimals like $1.414236\ldots$ or $3.1415926\ldots$. More precisely, we consider objects of the form

$$\pm \quad a_k \ldots a_1 a_0 . a_{-1} a_{-2} a_{-3} \ldots$$

where each digit a_i belongs to $\{0, 1, \ldots, 9\}$ and the sequence continues indefinitely to the right, although there are only a finite number of digits to the left of the decimal point. We think of such a decimal as representing the sum

$$a_k 10^k \ldots + a_1 10 + a_0 + a_{-1} 10^{-1} + a_{-2} 10^{-2} + a_{-3} 10^{-3} + \ldots$$

In this scheme, rational numbers end up being represented by **repeating** decimals, and conversely and repeating decimal is a rational number. There is a bit of ambiguity occasionally, since non-zero numbers ending in an infinite string of zeros can be rewritten as to end in an infinite string of 9's. Thus

$$.56 = .56000\ldots = .55999\ldots$$

both represent the fraction $\frac{14}{25}$. We could avoid this ambiguity by insisting that no non-zero decimals end in a string of zeros.

The set of all posssible decimals is a larger number system, also closed under addition, substraction, multiplication and division by non-zero numbers, called the **real number** system and denoted by $\mathbb{R}$. Since fractions can be written as decimals, $\mathbb{Q} \subset \mathbb{R}$.

All very familiar, right? Well, let's consider a simple question. How precisely do you add two decimal numbers? How for example do we compute $\sqrt{2} + \pi$, where we'll assume that the expansion of both $\sqrt{2}$ and π are given to us-perhaps in two (infinitely long) books, or better yet, by two black boxes (computer programs perhaps?) that for each number will input an integer n and output the n^{th} digit a_n .

*Norman Wildberger is a Pure Mathematician at the University of NSW.

In 6^{th} grade we learn to place one decimal above the other and add column by column, remembering to start on the right, carry to the left and (in case we are substracting) borrow from the left. Trouble is, there is **no place** to start at the right. The best it seems we can do is to chop off both decimals somewhere, compute the sum of the two truncations as in the example below and remove the possible ambiguity in the digit(s) at the tail by taking larger and larger truncations.

$$\begin{array}{rl} 1.41422136 & \ldots \\ + \quad \underline{3.1415926} & \ldots \\ 4.5558062 & \ldots \end{array}$$

In the example, it is clear that the final digit 2 we obtained is possibly temporary -it might end up being a 3 due to a later 'carry', or even a 4 perhaps? Could it end up being a 5? And exactly how far do we have to proceed with this algorithm before we can say with certainly what the a_{-7} digit (that is, where the 2 is presently) actually is? If you consider the possibility that our summands have 100 5^s and 4^s respectively following the 6^s, you see that it might be necessary to calculate to the -107^{th} digit before we can be certain about the -7^{th} digit.

In fact an unpleasant possibility arises -there might be an infinite string of such 5^s and 4^s (or any other combination of digits that result in an infinite row of 9^s in the sum) without us being in a position to predict it. Now if the two given numbers have decimal expansions which are the result of finite computer programs ($\sqrt{2}$ and π are of this kind) then we could conceivably study these programs and determine externally if and/or when such strings of 9^s occur. But we can prove that most irrational numbers do not have decimal expansions of this kind. This means that in general we have no way of knowing whether a given sequence of 9^s in a sum will carry on indefinitely. Thus it seems impossible for us to specify an algorithm to add two arbitrary real numbers.

If addition is not as straight-forward as we imagined, what about multiplication? Surely we should be at least able to multiply two fractions like $\frac{7}{9}$ and $\frac{2}{3}$ using decimals. Let's try. Writing the expansions of each, we get

$$\begin{array}{rl} .77777 & \ldots \\ \times \quad \underline{.66666} & \ldots \\ .46666 & \ldots \\ + \quad .04666 & \ldots \\ + \quad .00466 & \ldots \\ + \quad .00046 & \ldots \end{array}$$

⋮

The first decimal under the line represents the product $.7777\ldots \times .6$, which we can persuade ourselves is $.4666\ldots$ by employing our 'truncation algorithm' outlined above. The second line is similarly $.7777\ldots \times .06$, and so on. Now we are faced with the more delicate problem of adding up the resulting infinite stack of decimals. The problem of carrying is now increased– carrys will become larger and larger as we move right. Try it do you get the 'right answer' of $\frac{14}{27} = .518518518\ldots$?

Question. In general, how do we know this method actually works?

Question. Why is multiplication of decimals commutative? Why is it associative?

Addition and multiplication of decimals is subtle because carrying proceeds to the left and the infinite decimals go off to the right. Wouldn't it be easier if we had 'decimals' which went off to the **left instead**? That way, some of the problems just mentioned would evaporate.

Let's make it so. Declare a 'revercimal' to be a string like $\ldots 95141.3$ or $\ldots 6666.342$. More precisely, we consider objects of the form

$$\ldots a_3 a_2 a_1 a_0 . a_{-1} a_{-2} \ldots a_{-k}$$

where each digit a_i belongs to $\{0, 1, \ldots, 9\}$ and the sequence continues indefinitely to the left, although there are only a finite number of digits to the right of the decimal point. Note however a significant difference with the decimal situation–we do not defined negative reversimals). We would like to think of the above reversimal as representing the sum

$$\cdots + a_3 10^3 + a_2 10^2 + a_1 10 + a_0 + a_{-1} 10^{-1} + a_{-2} 10^{-2} + \ldots, + a_{-k} 10^{-k}$$

although such an infinite sum does not 'converge' in the usual sense. Euler, the great Swiss mathematician who lived in the 18^{th} century, had no particular qualms about working with such series–let us here follow his bold approach.

We will thus define addition, substraction and multiplication of revercimals exactly as we do for finite decimals: start on the right, carry to the left, and if substracting, borrow from the left. A few examples should suffice to get the hang of it...

1)

$$\begin{array}{rcccccccccc} & \ldots & 9 & 8 & 7 & 3 & 2 & 1 & . & 3 & 4 \\ + & \ldots & 2 & 0 & 4 & 6 & 9 & 5 & . & 1 & \\ \hline & \ldots & 1 & 9 & 2 & 0 & 1 & 6 & . & 4 & 4 \end{array}$$

2)

$$\begin{array}{cccccccccccc} & \ldots & 2 & 3 & 4 & 6 & 8 & 5 & 1 & . & 3 & \\ - & \ldots & 4 & 7 & 8 & 9 & 4 & 1 & 0 & . & 2 & 7 \\ \hline & \ldots & 7 & 5 & 5 & 7 & 4 & 4 & 1 & . & 0 & 3 \end{array}$$

3)

$$\begin{array}{cccccccccccc} & & & & \ldots & 7 & 7 & 7 & 7 & 7 & . \\ & & & \times & \ldots & 6 & 6 & 6 & 6 & 6 & . \\ \hline & & & & \ldots & 6 & 6 & 6 & 6 & 2 & . \\ & & & + & \ldots & 6 & 6 & 6 & 2 & 0 & . \\ & & & + & \ldots & 6 & 6 & 2 & 0 & 0 & . \\ & & & + & \ldots & 6 & 2 & 0 & 0 & 0 & . \\ & & & + & \vdots & & & & & & \\ \hline \ldots & 4 & 8 & 1 & 4 & 8 & 1 & 4 & 8 & 2 & . \end{array}$$

4)

$$\begin{array}{ccccccccccc} & & & & \ldots & 2 & 2 & 2 & 2 & 2 & 3 & . \\ & & & & \ldots & 3 & 3 & 3 & 3 & 3 & 4 & . \\ \hline & & & & \ldots & 8 & 8 & 8 & 8 & 9 & 2 & \\ & & & + & \ldots & 6 & 6 & 6 & 6 & 9 & & \\ & & & + & \ldots & 6 & 6 & 6 & 9 & & & \\ & & & + & \ldots & 6 & 6 & 9 & & & & \\ \hline \ldots & 4 & 8 & 1 & 4 & 8 & 1 & 4 & 8 & 2 & . & \end{array}$$

How simple these operations are compared to the corresponding calculations with decimals! No ambiguity or uncertainty, and, as we mentioned before, no negative revercimals either. Thus to find a revercimal α such that $\ldots 9514441.3 + \alpha = 0$, we perform the following

$$\begin{array}{ccccccccc} & \ldots & 9 & 5 & 1 & 4 & 1 & . & 3 \\ + & \ldots & a_4 & a_3 & a_2 & a_1 & a_0 & . & a_{-1} \\ \hline & \ldots & 0 & 0 & 0 & 0 & 0 & . & 0 \end{array}$$

and evaluate the $a_i'^s$ successively, starting from the right. (The answer is $\ldots 04848.7 = \alpha$) (Those of you who have never felt entirely happy with 'negative' numbers can breathe a sigh of relief.)

At this point, we do not know how to incorporate the rational numbers which are not finite decimals into $\tilde{\mathbb{R}}$. To do this, we employ a trick going back to Euler. He stated that for any x, the doubly infinite series

$$\ldots + x^3 + x^2 + x + 1 + x^{-1} + x^{-2} + \ldots$$

is 0. In our context, we may use and justify this as follows. Consider for a moment the 'doubly infinite' decimal

$$\alpha = \ldots 7777.7777\ldots$$

Since multiplying by 10 moves the decimal over, $10\alpha = \alpha$. Thus $\alpha = 0$ and so

$$\ldots 2223. = -\ldots 7777. = .7777\ldots = \frac{7}{9}.$$

Similarly for example, $\ldots 33334. = -\ldots 6666. = \frac{2}{3}$, and

$$\begin{aligned} \ldots 81481482. &= -\ldots 518518518 \\ &= .518518518\ldots \end{aligned}$$

We see that the example 4) worked out earlier is just the 'revercimal' form of the multiplication $\frac{7}{9} \times \frac{2}{3} = \frac{14}{27}$ that we computed earlier using the more dubious operations of decimal arithmetic. In fact Euler's 'trick' allows us to rewrite any rational as a revercimal. Here another example which should get the reader firmly on the road to reversimal arithmetic. To calculate $\frac{13}{99} \times \frac{8}{3}$, we write

$$\begin{aligned} \frac{8}{3} &= 2 + .6666\ldots = 2 - \ldots 66666. = 2 + \ldots 3334. \\ &= \ldots 3336. \\ \frac{13}{99} &= .131313\ldots = -\ldots 131313. = \ldots 6868687. \end{aligned}$$

Then the product is

$$\begin{array}{ccccccccccccc}
 & & \ldots & 6 & 8 & 6 & 8 & 6 & 8 & 6 & 8 & 7 & . \\
 & & \ldots & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 6 & . \\
\hline
 & & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 2 & . \\
 & & 0 & 6 & 0 & 6 & 0 & 6 & 0 & 6 & 1 & 0 & . \\
 & & 6 & 0 & 6 & 0 & 6 & 0 & 6 & 1 & 0 & 0 & . \\
 & & 0 & 6 & 0 & 6 & 0 & 6 & 1 & 0 & 0 & 0 & . \\
 & & 6 & 0 & 6 & 0 & 6 & 1 & 0 & 0 & 0 & 0 & . \\
 & & 0 & 6 & 0 & 6 & 1 & 0 & 0 & 0 & 0 & 0 & . \\
 & & 6 & 0 & 6 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & . \\
 & & 0 & 6 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . \\
 & & 6 & 1 & & & & & & & & & \\
 & & 1 & & & & & & & & & & \\
\hline
\ldots & 4 & 9 & 8 & 3 & 1 & 6 & 4 & 9 & 8 & 3 & 2 & .
\end{array}$$

or $-\ldots 50168350168. = .350168350168\ldots$ which is in fact the correct answer $\frac{104}{297}$.

Question. Does this really work, and if so why?

The problem of division, or equivalently of inverting revercimals still remains. Lets try to invert $\alpha = \ldots 73653$. That is we look for a reversimal $\beta = \ldots a_2 a_1 a_0 . a_{-1} \ldots a_{-k}$ such that $\ldots \alpha\beta = 1 = \ldots 00 \ldots 1$. This is accomplished in this case by setting $k = 0$, and solving for a_0, a_1, a_2 recursively. We first determine $a_0 = 7$ since the right most digit of the product will be a 1, then proceed, choosing each digit successively so that the final sum is $\ldots 00001$. Hopefully the following will make this clear.

$$
\begin{array}{cccccccc}
 & \ldots & 7 & 3 & 6 & 5 & 3 & . \\
\times & & & \ldots & 3 & 1 & 7 & . \\
\hline
 & & 1 & 5 & 5 & 7 & 1 & . \\
+ & & 3 & 6 & 5 & 3 & 0 & . \\
+ & \ldots & 9 & 5 & 9 & 0 & 0 & . \\
\vdots & & & & & & & \\
\hline
 & & & \ldots & 0 & 0 & 1 & .
\end{array}
$$

What makes this work is that for any digit d, we may find a digit m, such that the right most digit of $3m$ is d. In other words, $3m \equiv d (\bmod 10)$ has a solution for any d. This number theoretic fact does no longer hold if we replace 3 with a number that has a common factor with 10. For example, $2m \equiv d (\bmod 10)$ does not have a solution for some values of d. This inconvenience is solely due to our choice of 10 as the base of our number system; if we replaced it with a prime p then the above argument would show that all non zero reversimals in base p would be invertible. That would then be a pleasant situation indeed. The number system so obtained we could denote by $\tilde{\mathbb{R}}_p$, or perhaps by $\mathbb{Q}_p$.

However, other revercimals are also invertible. All repeating revercimals are, since they are rationals, so is $\ldots 02.$ since $\ldots 002. \times \ldots 00.5 = \ldots 0001$.

Question Are all non-zero revercimals invertible?

One of the most useful concepts in dealing with numbers is that of *size*. For the real numbers, the absolute value of a number plays this role. It has the following basic properties.

$A1)$ $|x + y| \leq |x| + |y|$ for all $x, y \in \mathbb{R}$

$A2)$ $|xy| = |x||y|$ for all $x, y \in \mathbb{R}$

To a first approximation, the size of a decimal x is 10^k where k is the place of the left most non zero digit of x. We now define a notion of absolute value for the reversimal numbers $\tilde{\mathbb{R}}$ using a variant of this idea. If α is a reversimal with the rightmost non zero digit in the kth spot, we declare

$$|\alpha|_{10} = 10^{-k}.$$

Thus for example $|320|_{10} = \frac{1}{10}$ and $|\ldots 243.12|_{10} = 100$. Thus the value $|\alpha|_{10}$ is always a power of 10, and hence rational. It is then not hard to check that

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