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# RAMSEY NUMBERS AND THE DESTRUCTIVE DEMON

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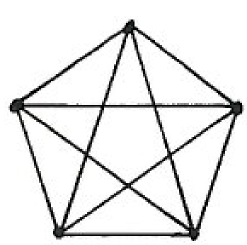
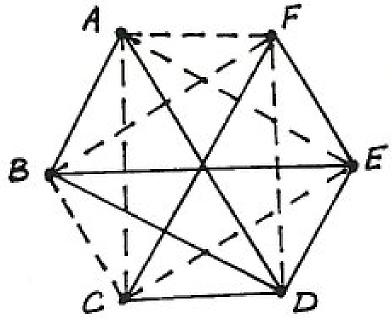
Combinatorial mathematics or combinatorics, as it is often called, is concerned with problems of arrangement and counting. Although such problems were often studied for their amusement or aesthetic value, they now have an important place in modern science, especially with its new emphasis on computers and technology. You have probably solved some kind of combinatorial problem at some time in your mathematical studies. For example, in how many ways can 10 people stand in a line? There are 10 choices for the first place, 9 for second, 8 for the third and so on, so there are $10.9.8\cdots 1 = 3,628,800$ possible arrangements. This is a simple example of enumeration or counting. Combinatorial maths also involves arrangement problems. For example, suppose we have the set of numbers $S = \{1, 2, 3\cdots 7\}$. Can we form 7 subsets of size 3 with elements from $S$ such that each pair of elements is contained in exactly one subset?

A bit of trial and error leads to the following arrangement

$$\{1,2,4\},\ \{2,3,5\},\ \{3,4,6\},\ \{4,5,7\},\ \{5,6,1\},\ \{6,7,2\},\ \{7,1,3\}.$$

Such a collection of sets is called a block design (technically a $(7,7,3,3,1)$-design). Block designs are extremely important in statistics as well as being of great interest in their own right. One of the most difficult areas of combinatorial mathematics (if not of all mathematics) is known as **Ramsey Theory**. Although difficult, it is possible for you to get a glimpse of what it is about. The basic ideas were put forward by the English logician Frank P. Ramsey and published in 1930. (Ramsey unfortunately died at the early age of 26; his brother became the Archbishop of Canterbury.) The easiest way to understand the key idea is via a little graph theory. Arrange $n$ dots (called **vertices**) in a circle and connect every dot to every other dot by a line (called an **edge**). We denote the corresponding diagram (called a **graph**) by the symbol $K_n$. For example, $K_5$, is drawn adjacent. Now, suppose we have $K_6$ and we colour each of its edges using the colours red

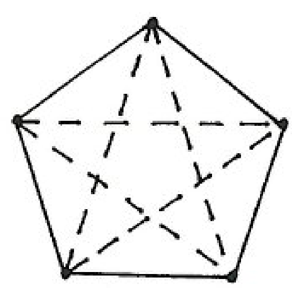





or blue. I claim that somewhere in the picture is a triangle that has 3 red sides or one that has 3 blue sides regardless of how we colour the edges. This is a favourite competition problem. For example, in the accompanying diagram, $\triangle ABD$ is red. (I use dotted lines for blue and a solid line for red.)

Here is the general argument: Choose a vertex, say $A$. Now $AB, AC, AD, AE$ and $AF$ are coloured either red or blue, so at least three of them are red or three of them are blue. Suppose for instance $AB, AD, AE$ are red. Now if $BD$ is red then $\triangle ABD$ is a red triangle, and similarly if $DE$ or $BE$ are red then $\triangle ADE$ or $\triangle ABE$ are red. If none of these edges are red then they are all blue so $\triangle BDE$ is blue. Thus the coloured graph always contains a red or a blue triangle.

Now we can describe a triangle as $K_3$, and state the above result as: If the edges of $K_6$ are coloured red or blue then a red $K_3$ or a blue $K_3$ must be found inside the graph. It is clear that if we took $K_7$, $K_8$, etc, then the same thing must happen. For $K_5$ this is not true, as is shown in the diagram below.



Six is the smallest number that works. We call this the Ramsey number $R(3,3)$. In general, $r = R(n,m)$ is the smallest integer such that if the edges of $K_r$ are coloured red or blue then the resulting coloured graph must have a red $K_n$ or $K_m$ or a blue $K_n$ or $K_m$ regardless of the colouring.

Suppose we wanted to know the Ramsey number $R(4,4)$ and had a computer (and a good programmer) on hand. Could we evaluate the number?

Well, we could do this by checking **all** possible configurations. The value of $R(4,4)$ is, in fact, 18 so we would need to look at all the graphs up to $K_{18}$ and show that $K_{18}$ has a red $K_4$ or a blue $K_4$ regardless of how we colour the edges, but that there were colourings of $K_{17}$ that do not contain a red or blue $K_4$. Now $K_n$ has $\frac{n(n-1)}{2}$ edges (why?) and each edge can be coloured red or blue, so there are $2^{\frac{n}{2}(n-1)}$ possible colourings to check.

For $K_{18}$ alone, this gives about $10^{46}$ graphs to check. This number could be reduced by symmetry arguments and possibly a powerful modern computer could then check all the cases, but this is not possible for $R(5,5)$, whose value is currently unknown. Below is shown a table of the current state of knowledge. A number in the centre of a square is the known exact value. A number at the top of a square is lower bound and a number at the top of a square is an upper bound for the Ramsey number.

| $mn$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 3 | 6 | 9 | 14 | 18 | 23 | 28 | 36 | 40<br>43 | 46<br>51 | 51<br>60 | 60<br>69 | 66<br>78 | 73<br>89 |
| 4 | | 18 | 25 | 35<br>41 | 49<br>62 | 53<br>85 | 69<br>116 | 80<br>151 | 93<br>191 | 98<br>238 | 112<br>291 | 119<br>349 | 128<br>417 |
| 5 | | | 43<br>49 | 58<br>87 | 80<br>143 | 95<br>216 | 114<br>317 | 445 | | | | | |
| 6 | | | | 102<br>165 | 300 | 497 | 784 | 1180 | | | | | |
| 7 | | | | | 205<br>545 | 1035 | 1724 | 2842 | | | | | |
| 8 | | | | | | 282<br>1874 | 3597 | 6116 | | | | | |
| 9 | | | | | | | 565<br>6680 | 12795 | | | | | |
| 10 | | | | | | | | 789<br>23981 | | | | | |

Many of the results in the table were obtained by intricate mathematical arguments (generally similar to the argument we used to show $R(3,3) = 6$) rather than straight number crunching. There are also formulae such as $R(p,q) \leq R(p-1,q) + R(p,q-1)$* for integers $p, q > 2$ which are used to get some of the upper bounds (e.g. the bound for $R(9,10)$ in the table is arrived at using this).

There is an old joke, due to the famous mathematician Paul Erdös, which says:

Suppose the world were about to be destroyed by a powerful demon, who demanded that we tell it the Ramsey number $R(5,5)$. What should we do? The answer is, to get all the mathematicians of the world to apply themselves day and night to finding $R(5,5)$. Now suppose that the demon wanted **both** $R(5,5)$ and $R(6,6)$. What should we do? The answer is, to get all the scientists of the world to apply themselves day and night to work out how to destroy the demon! The point of the joke is, of course, that with a huge amount of

* **Editor's Comment**: This inequality was proved by George Szekeres over 60 years ago in a joint paper he published with Paul Erdös. George is Emeritus Professor of Pure Mathematics at UNSW and is a member of **Parabola's** editorial board.

effort it seems just possible to find $R(5,5)$, but that the determination of $R(6,6)$ probably lies far off in the future.

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Continued from p.16

$B1)$ $|\alpha+\beta| \le \min(|\alpha|+|\beta|)$ for all $\alpha, \beta \in \tilde{R}$

$B2)$ $|\alpha\beta| \le |\alpha||\beta|$ for all $\alpha, \beta \in \tilde{\mathbb{R}}$.

Note the differences between these properties and the former ones. The reason that we don't get equality in property B2) is again related to the fact that 10 is not a prime; consider for example taking absolute valued of the equation $2 \times 5 = 10$.

Property B1) has numerous curious consequences one of them is that any triangle formed from three reversimals is isosceles. Perhaps the reader can discover others.

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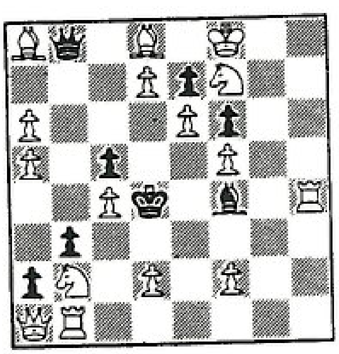
## A VERY SPECIAL CHESS PROBLEM

The following problem is due to L Yarosh. Its publication in the magazine **Shakhmatny v SSSR** in 1983 caused considerable suprise. The problem is of the 'white to play and mate in 4' type andhas a very remarkable solution, given on page 32.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| White: | Kf8 | Qa1 | Rb1 | Rh4 | Ba8 | Bd8 | Nb2 | Nf7 |
| | Pa5 | Pa6 | Pc4 | Pd2 | Pd7 | Pe6 | Pf2 | Pf5 |
| Black: | Kd4 | Qb8 | Bf4 | Pa2 | Pb3 | Pc 5 | Pe7 | Pf6 |



*Mate in 4*
*L. Yarosh*
*1st Prize, Shakhmatny 1983*