SOLUTIONS TO PROBLEMS 949-956

Q. 949 (a) We have a collection of numbers, each of which is either zero or one. Not all of the numbers are the same, and the total number of elements in the collection is prime. It is permitted to choose any two or more of the numbers (but not the whole collection) and replace each of them by the average of the chosen numbers. Show that no matter how often we perform this replacement operation we shall never reach a situation in which all numbers in the collection are the same.

(b) For this question recall that the *geometric mean* of two positive numbers x and y is defined to be $\sqrt{xy}$.

A collection of 1995 numbers consists of 1994 twos and a one. It is permitted to choose any two numbers from the collection and replace each of them by the geometric mean of the two. Is it possible by repeating this operation to obtain a collection in which all 1995 numbers are the same?

ANS. (a) Let the number of elements in the collection be p, and let the number of ones in the initial collection be k. If all the numbers in the collection become equal, they must be $\frac{k}{p}$. Note that since $1 \leq k \leq p-1$, this fraction is in lowest terms. However, all the numbers in the original collection have denominator 1, which is not divisible by p; and if we average n fractions with denominators not divisible by p we obtain

$$\frac{\frac{a_1}{b_1} + \cdots + \frac{a_n}{b_n}}{n} = \frac{(a_1 b_2 \cdots b_n) + \cdots + (b_1 \cdots b_{n-1} a_n)}{n\ b_1 \cdots b_n}.$$

Here p is not a factor of n since $1 < n < p$; and p is not a factor of $b_1, \cdots, b_n$, by assumption; so p is not a factor of the denominator of this new fraction. Thus we can never obtain the fraction $\frac{k}{p}$ in which the denominator is divisible by p.

(b) For each number a in the collection define a number

$$b = \frac{\log a}{\log 2}.$$

The collection of all values of b consists initially of 1994 ones and a zero. Since

$$\frac{\log \sqrt{aa'}}{\log 2} = \frac{1}{2}\left(\frac{\log a}{\log 2} + \frac{\log a'}{\log 2}\right)$$

we see that taking the geometric mean of two numbers in the collection of as corresponds to taking the (ordinary) average of two numbers in the collection of bs. Thus, if we could use the geometric mean to make all the as equal, we could use ordinary averaging to make all the bs equal. But from the School Mathematics Competition solutions (last issue) we know that this is impossible.

Q. 950 Show that any triangle of one of the following types can be dissected into three isosceles triangles:

(a) acute-angled triangles;

(b) triangles with at least one 45° angle;

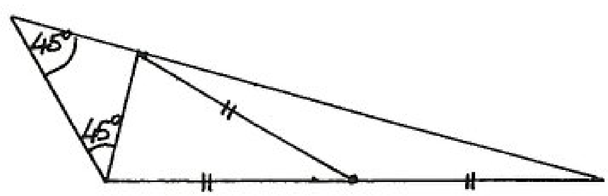
(c) triangles with one angle five times another;

(d) triangles with one angle six times another;

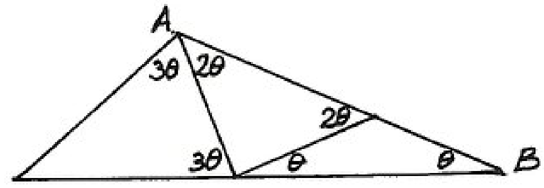
(e) triangles with one angle seven times another.

ANS. (a) Join each vertex to the circumcentre of the triangle.

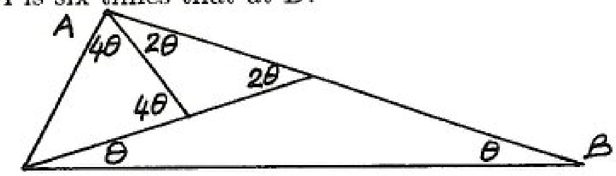
(b) Draw the perdendicular bisector of the shorter of the two sides adjacent to the 45° angle, thus creating a right-angled isosceles triangle; then divide the other right-angled triangle by bisecting its hypotenuse.



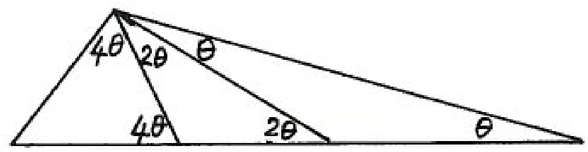
(c) As shown below, where the angle at A is five times that at B.



(d) The angle at A is six times that at B.



(e) The angle at A is seven times that at B.

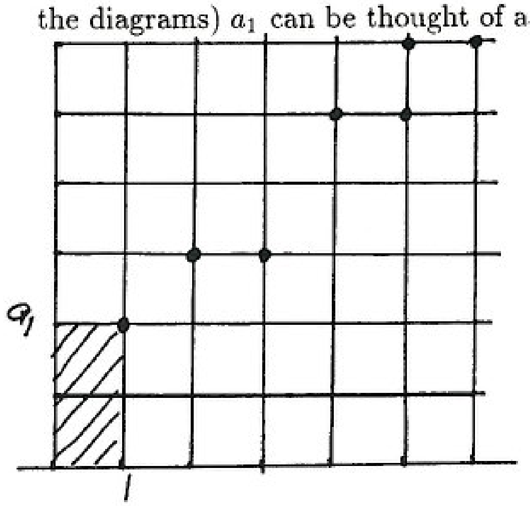


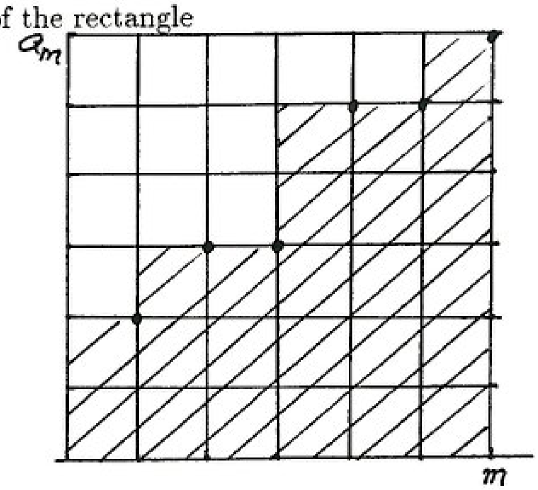
Martin Jenkins also solved the above problem.

Q. 951 Let $a_1, a_2, a_3, \ldots$ be a sequence of positive integers with the property that $a_1 \leq a_2 \leq a_3 \leq \cdots$. Define a second sequence $b_1, b_2, b_3, \ldots$, where b_k is the number of terms among $a_1, a_2, a_3, \ldots$ which are less than k. Prove that for any positive integer m, if n denotes the value a_m then

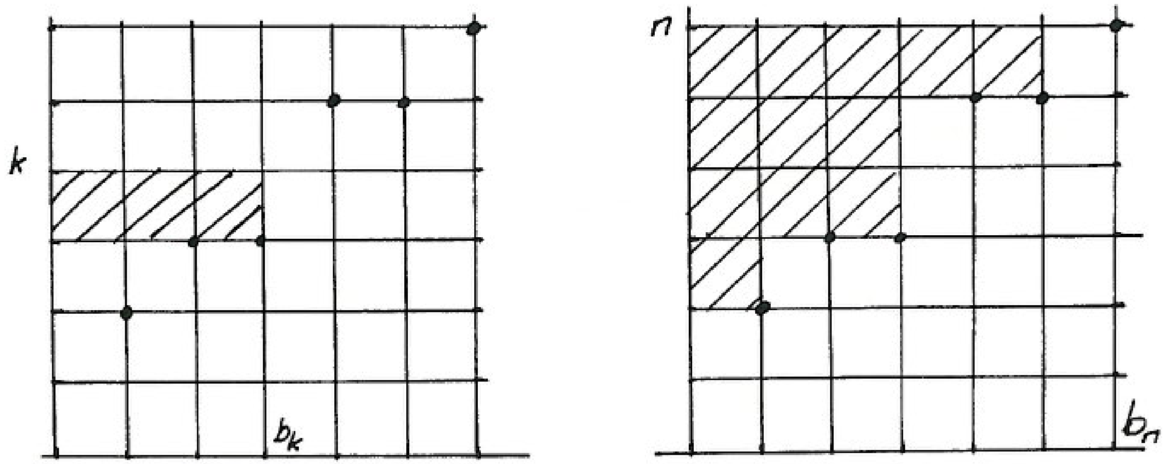
$$(a_1 + a_2 + \cdots + a_m) + (b_1 + b_2 + \cdots + b_n) = mn .$$

ANS. Plot the points $(1,\ a_1)$, $(2,\ a_2)$, $\cdots, (m,\ a_m)$ on a grid of unit squares. Then (see the diagrams) a_1 can be thought of as the area of the rectangle





defined by $0 \leq x \leq 1,\ 0 \leq y \leq a_1$; and the entire sum $a_1 + a_2 + \cdots + a_m$ will be the area shaded in the second figure. Similarly b_k will be the area of a rectangle $k-1 \leq y \leq k,\ 0 \leq x \leq b_k$, and the sum $b_1 + b_2 + \cdots + b_n$ will be the second shaded area below.



It can be seen that the two regions fit together without overlap to form a complete $m \times n$ rectangle, and so

$$(a_1 + a_2 + \cdots + a_m) + (b_1 + b_2 + \cdots + b_n) = mn.$$

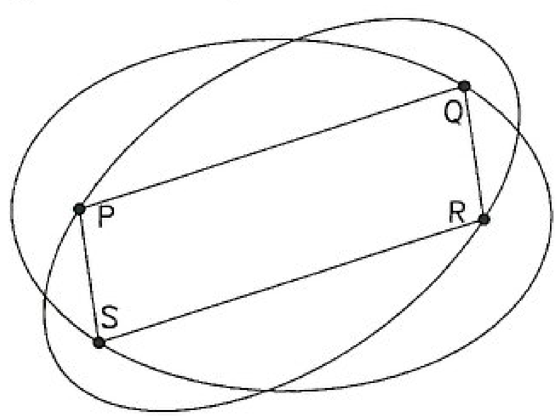
Comment. This result can also be proved by mathematical induction.

Q. 952 Each of two ellipses passes through the two foci of the other. Prove that

(a) the four foci lie at the vertices of a parallelogram;

(b) if the focal lengths of the two ellipses are equal, then the ellipses are congruent.

ANS. (a) We use the fact that for any particular ellipse the sum of the distances from the two foci to a point on the ellipse is constant



Applying this fact to the ellipse with foci P and R we have

$$PQ + QR = PS + SR;$$

and from the other ellipse

$$PQ + PS = QR + SR.$$

Adding these equations and simplifying gives $PQ = SR$; and hence $PS = QR$. Therefore the quadrilateral $PQRS$ has both pairs of opposite sides equal, and is a parallelogram.

(b) Hence the "distance constants" $PQ + QR$ and $QR + SR$ of the two ellipses are equal; and if the focal lengths are also equal then the ellipses are congruent.

Q. 953 Let a and b be unequal rational numbers. Show that

(a) if a and b are positive and $\sqrt{a} - \sqrt{b}$ is rational, then $\sqrt{a}$ and $\sqrt{b}$ are rational;

(b) if $\sqrt[3]{a} - \sqrt[3]{b}$ is rational, then $\sqrt[3]{a}$ and $\sqrt[3]{b}$ are rational.

ANS. (a) We have

$$(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b}) = a - b$$

and so

$$\sqrt{a} + \sqrt{b} = \frac{a - b}{\sqrt{a} - \sqrt{b}}$$

which is rational since it is given that a, b and $\sqrt{a} - \sqrt{b}$ are rational (and $\sqrt{a} - \sqrt{b} \neq 0$). Hence

$$\sqrt{a} = \frac{(\sqrt{a} + \sqrt{b}) + (\sqrt{a} - \sqrt{b})}{2}$$

is rational, being the average of two rational numbers; and likewise $\sqrt{b}$ is rational.

(b) Write $\alpha = \sqrt[3]{a}$, $\beta = \sqrt[3]{b}$, $k = \alpha - \beta$ and $\ell = \alpha\beta$. It is given that a, b and k are rational. Also

$$\begin{aligned} k^3 &= \alpha^3 - 3\alpha^2\beta + 3\alpha\beta^2 - \beta^3 \\ &= \alpha^3 - 3(\alpha - \beta)\alpha\beta - \beta^3 \\ &= a - 3k\ell - b \end{aligned}$$

and so

$$\ell = \frac{a - b - k^3}{3k}$$

which is also rational (noting tht $k \neq 0$). Eliminating β from the equations

$$\alpha - \beta = k, \quad \alpha\beta = \ell$$

yields

$$\alpha^2 - k\alpha - \ell = 0.$$

Now think of α as a variable. By long division of polynomials,

$$\alpha^3 - a = (\alpha^2 - k\alpha - \ell)(\alpha + k) + (k^2 + \ell)\alpha + (k\ell - a).$$

However $\alpha^3 - a = 0 = \alpha^2 - k\alpha - \ell$ and so

$$(k^2 + \ell)\alpha + (k\ell - a) = 0.$$

Finally,

$$\begin{aligned} k^2 + \ell &= (\alpha - \beta)^2 + \alpha\beta \\ &= \alpha^2 - \alpha\beta + \beta^2 \\ &= \frac{1}{4}(2\alpha - \beta)^2 + \frac{3}{4}\beta^2 \\ &\neq 0, \end{aligned}$$

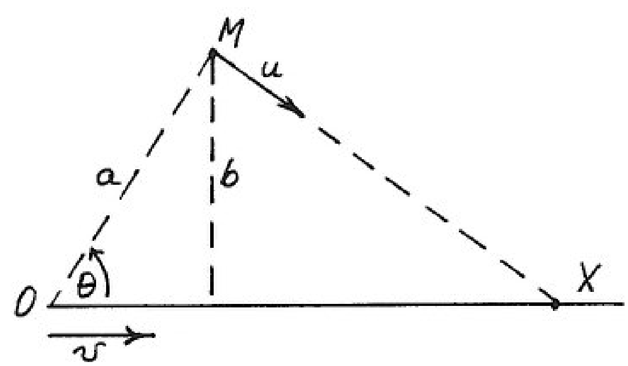
so

$$\alpha = \frac{a - k\ell}{k^2 + \ell}$$

which is rational since we know that a, k and ℓ are rational; and $\beta = \alpha - k$ is also rational.

Q. 954 A cyclist sets off from a point 0 and rides with constant velocity v along a straight highway. A messenger, who is at a distance a from point 0 and at a distance b from the highway wants to deliver a letter to the cyclist. What is the minimum velocity with which the messenger should run in order to achieve this outcome assuming she starts running at the same time the cyclist leaves 0?

ANS.



Suppose the cyclist and messenger meet at point X after time t. (We assume $b \neq 0$, leaving the case $b = 0$ as an exercise.) From the problem we can assume that the messenger runs directly to X along the straight line MX at some constant speed u, say. We therefore have $OX = vt$, $MX = ut$ and $OM = a$ so that, setting $\theta = \angle MOX$,

$$u^2t^2 = a^2 + v^2t^2 - 2avt\cos\theta \qquad \text{by the cosine rule.}$$

$$\therefore u^2 = \frac{a^2}{t^2} - 2a\frac{v}{t}\cos\theta + v^2$$

$$= \left(\frac{a}{t} - v\cos\theta\right)^2 + v^2\sin^2\theta.$$

To minimise this sum of squares we choose t so that $\frac{a}{t} - v\cos\theta = 0$ i.e. $t = \frac{a}{v\cos\theta}$. Since t must be positive we can do this provided $0 < \theta < \frac{\pi}{2}$ and we obtain a minimum speed $u_{\min} = v\sin\theta = v\frac{b}{a}$. When $\frac{\pi}{2} \leq \theta < \pi$ it follows that $-v\cos\theta$ is positive and we can see that

$$u^2 = \left(\frac{a}{t} - v\cos\theta\right)^2 + v^2\sin^2\theta > v^2\cos^2\theta + v^2\sin^2\theta = v^2$$

for all choices of t. As $t \to \infty$ it follows that $u \to v$ so that, provided $u > v$, the messenger can overtake the cyclist given enough time. (Of course if $u \leq v$ the messenger can never reach the cyclist.)

Q. 955 Prove that the polynomial $x^{44} + x^{33} + x^{22} + x^{11} + 1$ is divisible by the polynomial $x^4 + x^3 + x^2 + x + 1$.

ANS. By using long division **Martin Jenkins** discovered that $x^{44} + x^{33} + x^{22} + x^{11} + 1$ is the product of $x^4 + x^3 + x^2 + x + 1$ and $(x^{40} - x^{39}) + (x^{35} - x^{34}) + (x^{30} - x^{28}) + (x^{25} - = x^{23}) + x^{20} - (x^{17} - x^{15}) - (x^{12} - x^{10}) - (x^6 - x^5) - (x - 1)$.

He suggested the above bracketing as an aid to verification, e.g.,

$$(x^4 + x^3 + x^2 + x + 1)(x^{40} - x^{39}) = x^{44} - x^{39}$$

Maria Jenkins began by factoring $x^{55} - 1$ in two different ways:

$x^{55} - 1 = (x^{11} - 1)(x^{44} + x^{33} + x^{22} + x^{11} + 1)$ [This is best seen by substituting $a = x^{11}$.]

$$= (x - 1)(x^{10} + x^9 + \cdots + 1)(x^{44} + x^{33} + \cdots + 1) \qquad (*)$$

and

$$x^{55} - 1 = (x^5 - 1)(x^{50} + x^{45} + \cdots + 1)$$

$$= (x - 1)(x^4 + x^3 + \cdots + 1)(x^{50} + x^{45} + \cdots + 1) \qquad (\dagger)$$

Maria wanted to say that $x^4 + x^3 + \cdots + 1$ must therefore divide $x^{10} + x^9 + \cdots + 1$ or $x^{44} + x^{33} + \cdots + 1$.

So that it must divide the 2nd polynomial since it does not divide the first. However Martin correctly protested that one couldn't say this without extra argument. One "elementary" way of proving this is as follows. From (*) and (†) we know

$$(x^{10} + x^9 + \cdots + 1)(x^{44} + x^{33} + \cdots + 1) = (x^4 + x^3 + \cdots + 1)(x^{50} + x^{45} + \cdots + 1)$$

Now write

$$\begin{aligned} x^{10} + x^9 + \cdots + 1 &= (x^{10} + x^9 + \cdots + x^6) + (x^5 + x^4 + \cdots + x) + 1 \\ &= x^6(x^4 + x^3 + \cdots + 1) + x(x^4 + x^3 + \cdots + 1) + 1 \\ &= (x^6 + x)(x^4 + x^3 + \cdots + 1) + 1 \end{aligned}$$

$$\therefore (x^6 + x)(x^4 + x^3 + \cdots + 1)(x^{44} + x^{33} + \cdots + 1) + (x^{44} + x^{33} + \cdots + 1)$$

$$= (x^4 + x^3 + \cdots + 1)(x^{50} + x^{45} + \cdots + 1)$$

$$\therefore x^{44} + x^{33} + \cdots + 1 = (x^4 + x^3 + \cdots + 1)\{(x^{50} + x^{45} + \cdots + 1)$$

$$-(x^6 + x)(x^{44} + x^{33} + \cdots + 1)\}$$

This imples $x^4 + x^3 + \cdots + 1$ divides $x^{44} + x^{33} + \cdots + 1$.

There is a third method of attacking this problem which relies on knowledge of complex numbers. Again notice that

$$(x - 1)(x^4 + x^3 + \cdots + 1) = x^5 - 1$$

Therefore the roots of $x^4 + x^3 + \cdots + 1 = 0$ are the 4 complex roots of $x^5 = 1$. These roots are $w_k = \cos\frac{2\pi k}{5} + i\sin\frac{2\pi k}{5}$ for $k = 1, 2, 3, 4$ and it follows that

$$x^4 + x^3 + \cdots + 1 = (x - w_1)(x - w_2)(x - w_3)(x - w_4).$$

Now remember that each root w_k satisfies $w_k^5 = 1$

$$\therefore w_k^{44} + w_k^{33} + w_k^{22} + w_k^{11} + 1 = w_k^4 + w_k^3 + w_k^2 + w_k + 1 = 0$$

i.e., each w_k is a root of $x^{44}+x^{33}+\cdots+1=0$ so that $x-w_k$ is a factor of $x^{44}+x^{33}+\cdots+1$. This means that

$$x^4+x^3+\cdots+1 \quad \text{divides} \quad x^{44}+x^{33}+\cdots+1.$$

Q. 956 A party of four hikers who walk at 6kph and one motor cyclist who travels at 30kph leave town A simultaneously on a journey to town B, which is 45kms from A.

The motor cyclist can carry one passenger and carries each hiker a part of the journey and then returns for the others in turn. Find the minimum time required for the whole party to reach town B, and find how far each pedestrian has to walk.

ANS. First observe that if the whole party is to reach town B in minimum time then all hikers (h_1, h_2, h_3 and h_4) and the motor cyclist (m) must reach B together, and conversely that if they do reach B together then they will have completed their journey in minimum time. This means that all hikers must get a lift from m for the same length of time. For the time being let's forget that they wish to reach B together and ask when, and where, they will then meet up again if each hiker gets a lift **for exactly 1 hour.**

In this case after 1 hour h_1 has travelled 30km, whilst h_2, h_3 and h_4 have travelled 6km. It then takes m just $\frac{24}{36}=\frac{2}{3}$ hour to return for h_2 since h_2 and m are approaching one another at the rate of 36km/h. After 1 more hour h_1 and h_2 are together. After another $1\frac{2}{3}$hours h_3 catches up with h_1 and h_2, and $1\frac{2}{3}$ hours after tht h_4 is reunited with h_1, h_2 and h_3. In total $1+1\frac{2}{3}+1\frac{2}{3}+1\frac{2}{3}=6$ hours have passed and the group has travelled $1\times 30+5\times 6=60$kms. If the group only wanted to scale 45kms then we scale down: each hiker gets a lift on the bike for $\frac{45}{60}=\frac{3}{4}$hr, in a journey that takes $6\times\frac{3}{4}=4\frac{1}{2}$hrs.

Martin Jenkins provided a more direct solution.

Imagine what happens from the point of view of h_1. Suppose h_1 travels a km on the bike and then a further $b$km before h_2 catches up. It follows that h_1 travels $(a+3b)$km before the other three catch up.

$$\therefore a=3b=45. \qquad (*)$$

On the other hand reference to a diagram makes it clear that m travels $(a-b)+a=(2a-b)$km in riding back to pick up h_2, and then catching up with h_1. In this time h_1

travels $b$km so that

$$2a - b = 5b \tag{†}$$

given that m is 5 times faster than h_1.

It follows from (∗) and (†) that $a = \frac{45}{2}$ and $b = \frac{15}{2}$ and the total time taken for the trip equals $\frac{3}{4} + \frac{15}{4} = 4\frac{1}{2}$hr.

SOLUTION TO CHESS PROBLEM.

Black has to guard the bishop of f4 to avoid an early mate.

This leads to the following fairly dull variations after the key 1. a7:

1. a7	Q×d8+	2. Kg7	Qc7	3. d8=Q+	Q×d8	4. Rf4×.
	Q×a8	2. R× f4+	Qe4	3. a8=Q	???	4. Qd5×.
	Qc7	2. B× c7	a× b1=Q	3. d8= Q×.		
	Qd6	2. Re1	Qe5	3. N×e5	f×e5	4. Re4×.
	Qe5	2. B× e7	Qd6	3. N×d6	Ke5	4. Nd3×.

The real point comes when the captures of the rook on b1 are analysed:

a×b1=Q	2. a× b8=Q	Q×b2	3. Q×b3	Qc3	Q×c3×.
		Qe4	3. Q×f4	Q×f4	4. R× f4×.

You have not seen anything much yet. Black can try for stalemates:

a×b1=R	2. a×b8=R	R×b2	3. R×b3	K×c4	Qa4×.
a×b1=B	2. a×$b8 = B$	Be4	3. B×f4	B??	Be3×.
a×b1=N	2. a×$b8 = N$	Nd2	3. Qc1	Ne4	Nc6×.

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