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THE FOUR COLOUR PROBLEM

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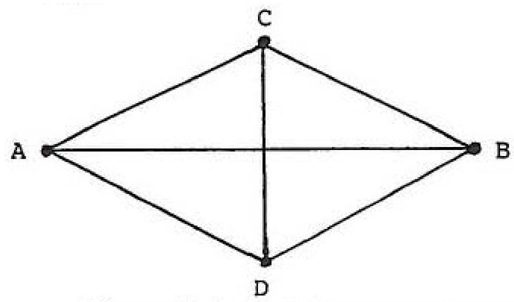
The introduction, this century, of computers into mathematics has certainly revolutionised the subject, and led to many new areas of mathematics previously unimagined. The use of computers in mathematical proofs, however, has been and remains a contentious issue. Can we 'trust' a computer to correctly prove a result? If millions of computations are required in the proof, how can we check them? Perhaps the machine made a mistake somewhere? This is not as improbable as it may first appear.

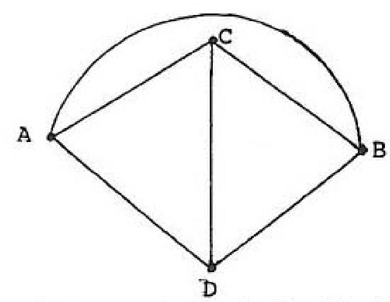
A mathematical proof is surely (or perhaps 'once was') a set of arguments whose validity can be checked at any time by anyone (who is sufficiently clever at mathematics). Is a 'proof' that relies on the computations of a machine (so many computations in fact that no human being could hope to check them all by hand), really to be regarded as a genuine mathematical proof? The old adage of the Greek philosopher Protagoras, that "**Man** is the measure of all things", takes on a new meaning in this context!

This philosophical problem first arose in 1976 when two mathematicians K. Appel and W. Haken announced they had a proof of a conjecture which came to be known as the Four Colour Theorem, which I will shortly describe. Their 'proof' involved the use of a computer to check the thousands of possible cases to which they had reduced the problem. I was once told that a certain professor of mathematics wept at the thought a computer being used to solve this difficult problem, but in any event, many in the mathematical community at the time were horrified by the thought.

The Four Colour Problem is concerned with graphs and I want to define what is meant by a planar graph. A graph is simply a collection of dots (called vertices) joined by lines (called edges) such that each edge joins exactly two (not necessarily different) vertices. Two vertices joined by one edge are said to be adjacent. For example in the diagram below the vertices A and B are adjacent. A planar graph is one which can be drawn without the edges crossing.

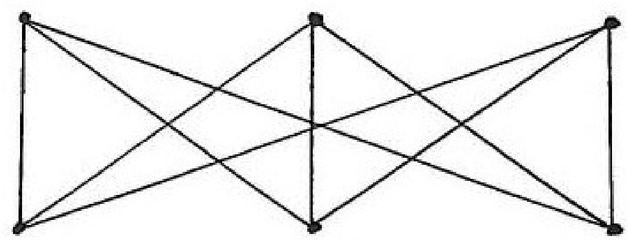
The graph below **is** planar since we can move the edge AB so that it does not cross any other edge. The graph drawn below is not planar since there is no way of redraw-



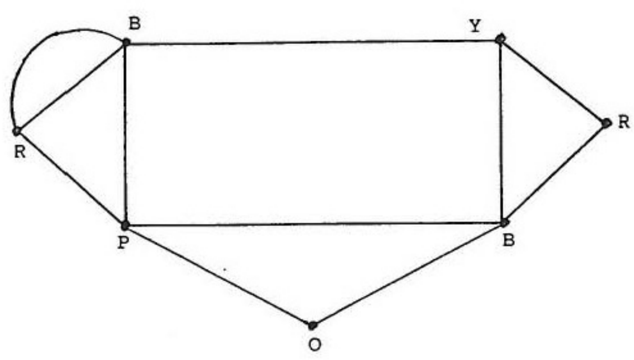


ing it without the edges crossing. (You should try redrawing it and see what happens.)

A colouring of a graph is simply an assignment of colours to the vertices so that

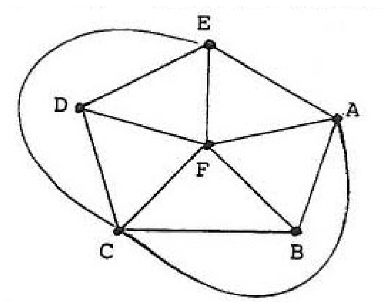


no two adjacent vertices receive the same colour. The graph below has been coloured using the five colours red (R), blue (B), yellow (Y), orange (O), and pink (P). The Four



Colour Theorem says that **every** planar graph can be coloured using only four colours. The truth of this was first surmised in 1852 and a 'proof', which turned out to be false, was given in 1879 by Kempe. Prior to the proof in 1976, it had been shown that every planar graph could be coloured using 6 colours and then it was shown that this number could be reduced to 5. The proof of the 5 colour theorem is not too hard, but does have a number of subcases to worry about. The proof for 6 colours is fairly easy and will give you an inkling as to how the other proofs work. To show the 6 colour result, we need a little preliminary result.

In a planar graph, in which every edge joins two different vertices (that is, there are no loops, where a vertex is joined to itself) and where any two vertices are joined by only one edge (such a graph is called a simple graph), then there must be a vertex whose degree is at most 5. The degree of a vertex is the number of edges it meets. So for example, in the following diagram, the degree of the vertices B and D is 3, the degree of A and E is 4 and the degree of C and F is 5. Assuming the truth of the statement given



above, we can now prove the 6 colour result by induction on the number of vertices. The result is clearly true for all planar graphs with six or fewer vertices, since we can colour each vertex and no two adjacent vertices receive the same colour. Suppose then that the result is true for all planar graphs with $n-1$ vertices. Clearly we need only consider simple planar graphs since any planar graph can be made simple by deleting only edges, which does not affect the colouring of vertices. Thus, by the preliminary result above, there must be a vertex V of degree less or equal to 5. If we delete this vertex for the moment, (and all the edges connecting it to other vertices) we have a graph with $(n-1)$ vertices which is colourable using the six colours, by assumption. Suppose then we have coloured **this** graph using 6 colours, and we now add back in the vertex V and all the edge we previously deleted. Since it is connected to at most 5 other vertices, I have (at least) one colour left to use on V which is different to the colours used on the other 5 vertices. Thus I have coloured the graph using only 6 colours and no two adjacent vertices have the same colour. The result is therefore true by induction.

The proof for the case of 5 colours is similar, but one must look at various cases. For 4 colours, a complicated extension of the arguments used in the 5 colour theorem resulted in reducing the problem to a finite, but very large number of cases, which were then checked, as I have said above, using a computer.

I recently came upon a similar example where a computer might be used in a proof. The problem was given to me by one of my students. Suppose we are given a positive integer and we calculate the sum of the fourth powers of its digits. For example 324 $\rightarrow 3^4+2^4+4^4 = 353$. Find all positive integers which remain unchanged by this process. Observe that if we take a six digit number (or larger) the process will give us a number with five digits or smaller. Thus, the problem is a finite one. We only need to check all the numbers with five digits or less. (In fact, a little thought reduces the number of cases considerably). We could then get a computer to check all these cases and the problem is solved. If you have a computer, then you might like to try this. One of the numbers is 8208, since 8208 = $8^4 + 2^4 + 0^4 + 8^4$. Although we can reduce the number of cases to be checked, it would still take a long time to check all the possibilities by hand. If we change the problem to 10th powers, then it would be impossible to check by hand and we would definitely need a computer. Does the use of a computer then, invalidate the work?

Just as the advent of genetic engineering has brought with it many social and moral problems to be considered and dealt with, so the introduction of computers has raised the difficult question as to what is meant by a 'proof' and the implications of this for mathematicians will continue to be controversial as we rely more and more on the machine.

NOT SO OBVIOUS.

$$\begin{array}{rr} & \text{F O R T Y} \\ + & \text{T E N} \\ + & \text{T E N} \\ \hline & \text{S I X T Y} \\ \hline \end{array}$$

This looks obvious. But can you replace the letters by digits in such a way that the resulting addition is still correct?

(Answer in the solution section)