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STRIP PATTERNS

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One of the main aims of Mathematics is to invent ideas and notation which will help understand the real world. For example, many years ago, the idea of number was invented to help people count and this idea was extended to what we now call real numbers to deal with measurement of distances and other things. Later, Descartes used the idea of an ordered pair of real numbers to simplify plane geometry, and Newton introduced the idea of a limit to deal with the concept of continuity.

The ideas of Mathematics have been successfully applied to many areas of Science (you may already have used some of them in Physics). But Mathematics has also been used in the study of some areas of art and design. One such area is in determining all the possible repeating patterns that are available for wallpaper and frieze designs. In both cases a basic shape (such as a flower) is repeated. In the case of wallpaper, it is repeated over the whole wall and, in the case of a frieze, the repetition is only along the strip. For example a frieze using the letter `V` looks like

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`... VVV ...`

So how can Mathematics determine all the different ways of constructing such a pattern? To answer this question, we need first to decide what we mean by different patterns. The above frieze (which was drawn in black) could equally well have been drawn in any other colour and clearly we would not want to have thought of it as a different pattern. Also the strip could have been turned over so that its top and bottom were reversed, resulting in the following "equivalent" frieze:

`... ΛΛΛ ...`

However there is a more subtle way in which we can construct an equivalent frieze. The symbol `Λ` looks very much like the letter `A` and so we would like to be able to say that the above frieze is equivalent to the following:

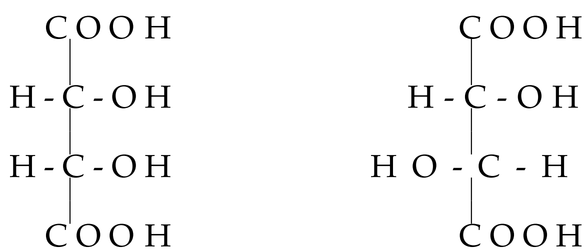
`... AAA ...`

Of course we cannot take this too far. We would like to be able to say that replacing the letter V (or Λ or A) by a completely different shape (such as the letter F) would yield a different frieze. The mathematical concept introduced to deal with this is that of **symmetry**. Symmetry occurs in many aspects of life such as:

1. Our own bodies have a certain degree of symmetry, although (as any left-handed person will tell you) they are not completely symmetric.

2. Our bodies, along with many other living things, contain proteins which are chains of amino-acids twisted in a helix. It is an interesting fact that these helices are always twisted in an anti-clockwise direction, and the desire for symmetry has led some Biologists to ask if there are other life-forms based on "clockwise" proteins.

3. Proteins themselves are made up of organic molecules which are arrangements of (mainly) carbon, oxygen and hydrogen atoms. Some of these arrangements are symmetric and some are asymmetric as illustrated by the following forms of tartaric acid:



4. Much Geometry, such as isosceles (and especially equilateral) triangles, is based on the symmetry of the figure. In fact, the ancient Greeks regarded a circle as a "perfect" figure, since it has an infinite number of axes of symmetry.

What then is symmetry? In everyday life, we think of a figure as being "symmetrical" if it has an axis of symmetry, i.e. if the figure is the same when reflected in a mirror placed along that axis.

E A H

Thus the letter E is horizontally symmetric, the letter A is vertically symmetric, and the letter H is both. Unfortunately, this concept is not completely adequate since the letter N has no axis of symmetry although it has a "symmetric" appearance. This is related to the fact that a reflection of H in a horizontal axis of symmetry followed by a reflection in its vertical axis is not a reflection. These facts are covered in the following definitions:

1. A **symmetry of an object** is a transformation of that object which

   (a) does not alter the object as a whole even if it moves some of its component parts (i.e. the transformed object is indistinguishable from the original);

(b) leaves all distances unchanged (i.e. if the points $P,\ Q$ are moved to the points $P',\ Q'$, then the distance between P' and Q' is the same as the distance between P and Q).

For example, a horizontal and a vertical reflection are both symmetries of the letter H, and the identity transformation id (which moves nothing) is a symmetry of any object.

2. If s, t are two symmetries of an object then the **product** st of s and t is the result of first transforming the object by s and then transforming the result by t.

   For example, the product of a horizontal reflection and a vertical reflection is a rotation through 180^o (usually called a **half-turn**).

   If you think about the idea of a symmetry of an object, it should be reasonably obvious that the identity transformation id is a symmetry of any object. Also, if s, t are two symmetries of an object, then st and s^{-1} are symmetries of the object (where $ss^{-1} = s^{-1}s$ is the identity transformation, i.e. s^{-1} 'undoes' s).

3. The set of all symmetries of an object is called its **symmetry group**.

   For example, the symmetry group of the letter F is $\{id\}$, and the only symmetries of the letter E are id and a horizontal reflection (you might like to find the symmetry groups of the remaining letters of the alphabet.)

   Also the symmetry group of a regular *plane* polygon with n edges consists of the n rotations through multiples of $360^o/n$, the symmetry group of a circle includes rotations through any angle, and the symmetry group of a regular polygonal *plate* with n edges consists of the above n rotations and the reflections in the n axes of symmetry.

It can be shown that the symmetry group of any bounded plane figure consists only of rotations and/or reflections.

In applying the theory of symmetry groups to unbounded figures such as lines, friezes and wallpaper designs, we need a third type of transformation: A **translation** is a transformation in which every point is moved some given (fixed) distance in a given direction.

For example, any translation along a line is a symmetry of that line and a translation along a frieze which takes one copy of the basic shape to another one is a symmetry of that frieze.

All the symmetries of a straight line can be found by regarding that line as a real number line. If a symmetry s moves the point 0 to the point a, then (since s must leave all distances unchanged) s must move an arbitrary point x to either of the points $a + x$ or $a - x$:

$0 \quad x \quad a - x \quad a \quad a + x$

Again because it must leave all distances unchanged, s cannot move one point x to $a+x$ and another point y to $a+y$. So the only possibilities are that s is a translation of a units along the line or a half-turn about the point $a/2$.

We can now find all the possible different strip patterns, where we say that

two shapes or patterns are different if and only if they have different symmetry groups.

So imagine a horizontal strip made up of a shape S repeated infinitely often along the strip. Any symmetry s of this strip must move S either to itself or to another shape S'.

In the first case (by what was said above about bounded plane figures and symmetries of a straight line) s must be a rotation or a reflection of S in one of its axes of symmetry. But s must not alter the strip as a whole and so must be id, a half-turn h (as in the case of the letter `Z`), a horizontal reflection r_H (as for the letter `E`) or a vertical reflection r_V (as for the letter `A`).

In the second case, suppose that t is the *translation* which moves S to S'. Then t^{-1} moves S' to S and so st^{-1} moves S to itself. Thus

$$\begin{aligned} st^{-1} &= id \quad h \quad r_H \quad \text{or} \quad r_V \\ s &= t \quad ht \quad r_H t \quad \text{or} \quad r_V t \end{aligned}$$

In the second and fourth of these cases it can be shown, using a method similar to the one used for the symmetries of a straight line, that s is a half-turn about a point or a reflection in a vertical line half-way between the shapes S and S'. In the third case, s is actually a new symmetry – called a **glide-reflection**. This may be thought of as a reflection in a mirror which slides along so quickly that the reflection is moved along the mirror. An example of a frieze which has a glide-reflection is

... LΓLΓ ...

So we have shown that there are five possible types of symmetry of a frieze: translations, half-turns, horizontal reflections, vertical reflections or glide-reflections. Each of these can be used to construct a different symmetry group, and so a different pattern on a strip. More symmetry groups can be constructed by using a combination of some of these five symmetries. However, after some experimenting, it is possible to show the results in the following table, where (for example) the first row says that the product of a translation (t) and a horizontal reflection (r_H) is a glide-reflection (g), the product of a translation and a glide-reflection is either a glide-reflection or a horizontal reflection, and the product of a translation and a translation, half-turn or vertical reflection is a translation, half-turn or vertical reflection respectively:

	t	h	r_H	r_V	g
t	t	h	g	r_V	g or r_H
h	h	t	r_V	r_H	r_V
r_H	g	r_V	id	h	t
r_V	r_V	r_H	h	t	h
g	g or r_H	r_V	t	h	t

Thus there are only seven possible patterns on a strip. Their symmetry groups consist of translations along the strip and

(a) nothing else	e.g.	...FFFF...
(b) half-turns	e.g.	...ZZZZ...
(c) a horizontal reflection	e.g.	...EEEE...
(d) vertical reflections	e.g.	...AAAA...
(e) glide-reflections	e.g.	...LΓLΓ...
(f) glide-reflections, half-turns and vertical reflections	e.g.	...VΛVΛ...
(g) all of the above	e.g.	...HHHH...

Puzzle

What does the following mean?

... DEEP DEEP ...

(Answer in the solution section)