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PROBLEM SECTION

You are invited to submit solutions to any or all of the following problems, accompanied by your name, school and year or form. Solutions of these problems will appear in the next issue of **Parabola**; your solution(s) may be used if they are received in time.

Q.966 Prove that

$$\binom{n}{1} - \frac{1}{2}\binom{n}{2} + \frac{1}{3}\binom{n}{3} - \cdots \pm \frac{1}{n}\binom{n}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\,,$$

where the final sign on the left hand side is $+$ if n is odd and $-$ if n is even.

Q.967 $ABCDE$ is a regular pentagon of side length 1, and X and Y are points on BC and ED such that $BX = p$ and $EY = q$. Find q in terms of p, given that the line XY divides the pentagon into two regions of equal area.

Q.968 The set of numbers $\{\,21, 24, 25, 29\,\}$ is given. It is permitted to multiply any two numbers from the set (multiplying one number by itself is also allowed) and place in the set the last two digits of the product. For example, since $21 \times 29 = 609$, we may put the number 09 into the set. By repeating this operation as often as desired, is it possible to obtain a set including the number 99?

Q.969 Five positive integers add up to 1996. Find all possible values of the greatest common divisor of the five numbers.

Q.970 Given a prime number p, find all integer solutions of the equation

$$\sqrt{x} - \sqrt{p} = \sqrt{y}\,.$$

Q.971

(a) Let m be a positive integer and n a positive integer with fewer than $m/9$ digits. Prove that if the sum of the digits of n and the sum of the digits of $2n$ are both multiples of m, then every digit of n is less than 5.

(b) Show also that if m is not a multiple of 3 then the above is true for all integers n having fewer than m digits.

Q.972

(a) Find all solutions in real numbers of the simultaneous equations

$$x^2 + y^2 = 13\,, \qquad x^3 + y^3 = 35\,.$$

(b) Show that for any given real numbers a and b, the equations

$$x^2 + y^2 = a\,, \qquad x^3 + y^3 = b$$

have at most two real solutions. (Interchanging x and y does not count as a new solution.)

Q.973 A triangle has n points located inside it, where n is a positive integer. These $n+3$ points (the three vertices of the triangle and the n interior points) are joined by straight lines in such a way that the original triangle is subdivided into smaller regions, every one of which is a triangle with vertices at three of the given $n+3$ points. Prove that it is possible to colour the lines red and blue so that for any two points P, Q of the given $n+3$, there is a blue path from P to Q and also a red path from P to Q.

Q.974 For any integers x, y an integer $x * y$ is defined by

$$x * y = \min\big(\,|x+y|\,,\,|x-y|\,\big)\,.$$

Calculate

(a) $(\cdots((1*2)*3)*\cdots*1995)*1996\,;$

(b) $1*(2*\cdots*(1994*(1995*1996))\cdots)\,;$

(c) $(-1996)*((-1995)*\cdots*(0*(1*(2*\cdots*(1994*(1995*1996))\cdots)))\cdots)\,.$