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SOLUTIONS TO PROBLEMS 957-965

Q.957 Solve the three simultaneous equations

$$\frac{ab}{a+b}=\frac{1}{2} \quad , \quad \frac{bc}{b+c}=\frac{1}{3} \quad , \quad \frac{ac}{a+c}=\frac{1}{9}.$$

ANS. Clearly a, b and c are non-zero. Therefore we may take the reciprocal of each equation and split up the fractions to obtain

$$\frac{1}{a}+\frac{1}{b}=2\,, \quad \frac{1}{b}+\frac{1}{c}=3\,, \quad \frac{1}{c}+\frac{1}{a}=9\,.$$

Adding the three equations and dividing both sides by 2 we have

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=7\,;$$

using this and the previous three equations we get

$$\frac{1}{a}=4\,, \quad \frac{1}{b}=-2\,, \quad \frac{1}{c}=5$$

and therefore

$$a=\frac{1}{4}\,, \quad b=-\frac{1}{2}\,, \quad c=\frac{1}{5}\,.$$

Q.958 By considering $(2+\sqrt{3})^n$ and $(2-\sqrt{3})^n$ show that the equation $x^2-3y^2=1$ has infinitely many integer solutions.

ANS. First note that

$$\big(2+\sqrt{3}\,\big)^n\big(2-\sqrt{3}\,\big)^n=\big(4-3\big)^n=1\,.$$

Expanding the first factor by the binomial theorem,

$$\begin{aligned}\big(2+\sqrt{3}\,\big)^n &= 2^n+n\,2^{n-1}\sqrt{3}+\binom{n}{2}\,2^{n-2}\,3+\binom{n}{3}\,2^{n-3}\,3\sqrt{3}+\cdots\\ &= x_n+y_n\sqrt{3}\,,\end{aligned}$$

where

$$x_n=2^n+\binom{n}{2}\,2^{n-2}\,3+\cdots \quad \text{and} \quad y_n=n\,2^{n-1}+\binom{n}{3}\,2^{n-3}\,3+\cdots\,.$$

Since binomial coefficients are integers we see that x_n and y_n are also integers. The binomial expansion of the second factor can be found simply by replacing $\sqrt{3}$ by $-\sqrt{3}$; thus

$$\left(2-\sqrt{3}\,\right)^n = x_n - y_n\sqrt{3}\,.$$

Substituting these expressions back into our first equation gives

$$\left(x_n + y_n\sqrt{3}\,\right)\left(x_n - y_n\sqrt{3}\,\right) = 1\;,$$

that is,

$$x_n^2 - 3y_n^2 = 1$$

for any n; and as noted above, x_n and y_n are integers. *Comment*. ¿From the above relations we can actually derive explicit formulae for the solutions of the equation,

$$x_n = \frac{\left(2+\sqrt{3}\,\right)^n + \left(2-\sqrt{3}\,\right)^n}{2} \quad \text{and} \quad y_n = \frac{\left(2+\sqrt{3}\,\right)^n - \left(2-\sqrt{3}\,\right)^n}{2\sqrt{3}}\,.$$

¿From above we know that x_n and y_n are integers despite the square roots in these formulae!

Q.959

(a) Suppose we are given 100 positive integers $x_1, x_2, \ldots, x_{100}$ between 1 and $100,000,000,000$. Prove that for at least two of these numbers, say x_i and x_j, the sum of the digits of x_i is the sum of the digits of x_j (e.g, if $x_i = 1416$ and $x_j = 222222$ then the sums of the digits are the same).

(b) Suppose we have $n+1$ real numbers where $n \geq 2$. Prove that for two of these numbers a and b we must have

$$0 \leq \frac{a-b}{1+ab} < \tan\frac{\pi}{n}.$$

ANS.

(a) The integer in the given range with the largest sum of digits is $99,999,999,999$, and the sum is 99. Therefore, there are only 99 possible different sums of digits for integers in the given range, and if we are given 100 integers then two of them must have the same sum of digits.

(b) Taking the inverse tangent (in radians!) of the $n+1$ given real numbers yields $n+1$ real numbers between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$. Since we have $n+1$ numbers all within an interval of length π, two of the numbers must be separated by less than $\dfrac{\pi}{n}$. Let α be the larger and β the smaller of these two numbers; then

$$0 \leq \alpha - \beta < \frac{\pi}{n}\,.$$

Now $\alpha = \tan^{-1} a$ and $\beta = \tan^{-1} b$, where a and b are two of the given numbers; since the graph of $y = \tan x$ is always increasing from 0 to $\frac{1}{2}\pi$ we may take the tangent of all three terms in the above inequality to give

$$0 \le \tan(\alpha - \beta) < \tan\frac{\pi}{n}\,.$$

On the other hand, we can apply the tangent-of-a-difference formula to the middle term to obtain

$$\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta} = \frac{a-b}{1+ab}$$

and this completes the proof.

Q.960 Suppose there are 499 students and 500 numbered lockers in a school. Initially all the lockers are open. The students line up and the first student closes every second locker, beginning with locker number 2. The next student then examines every third locker, beginning with locker number 3, and closes each locker if it is open whilst opening if it is closed. The remaining students continue this opening and shutting process. (The third student examines every fourth locker and so forth). Which lockers are open at the end of the procedure?

ANS. A locker changes from being open to closed, or *vice versa*, whenever it is examined by a student. So a locker, initially open, will be open at the end of the procedure if and only if it is examined an even number of times. But since the first student looks at every second locker, the second at every third and so on, locker n is opened or closed once for each factor of n other than 1, and hence the lockers which remain open at the end are those for which n has an even number of factors excluding 1, that is, an odd number of factors altogether. Now any factor a of n can be paired with the factor n/a, and collecting all these pairs gives an even number of factors. The only way to get an odd number is if for some a, the factors a and n/a are actually the same, that is, $n = a^2$. Thus at the end of the procedure, the open lockers are those with square numbers, that is, lockers $1, 4, 9, 16, 25, \ldots, 441$ and 484.

Q.961 Suppose a, b and c are roots of the equation $x^3 - x^2 - x - 1 = 0$.

(a) Show that a, b and c are distinct.

(b) Show that

$$\frac{a^{1000000} - b^{1000000}}{a-b} + \frac{b^{1000000} - c^{1000000}}{b-c} + \frac{c^{1000000} - a^{1000000}}{c-a}$$

is an integer.

ANS.

(a) A polynomial has two (or more) equal roots if and only if the polynomial and its derivative are simultaneously zero. By long division,

$$x^3 - x^2 - x - 1 = (3x^2 - 2x - 1)(\tfrac{1}{3}x - \tfrac{1}{9}) - \tfrac{8}{9}x - \tfrac{10}{9} \;;$$

if $x^3 - x^2 - x - 1$ and $3x^2 - 2x - 1$ are both zero then $x = -\frac{5}{4}$, and it is easy to check that this is not in fact a root of the equation. Thus the three roots are distinct.

(b) In fact

$$\frac{a^k - b^k}{a - b} + \frac{b^k - c^k}{b - c} + \frac{c^k - a^k}{c - a}$$

is an integer for any positive integer k. This can be proved simply by induction: for $k = 1, 2$ and 3 the expression is respectively 3, $2(a + b + c)$ and

$$(a^2 + ab + b^2) + (b^2 + bc + c^2) + (c^2 + ca + a^2) = 2(a + b + c)^2 - (ab + bc + ca) \;;$$

all three of these expressions are integers since the relations between roots and coefficients of a polynomial tell us in this case that $a + b + c = 1$ and $ab + bc + ca = -1$. Let $k \geq 1$ and assume that the given expression is known to be an integer when the exponent is $k, k + 1$ and $k + 2$. The cubic equation can be multiplied by x^k and rearranged to give the equation

$$x^{k+3} = x^{k+2} + x^{k+1} + x^k \;,$$

which is true for $x = a, b$ and c. Hence

$$\begin{aligned}\frac{a^{k+3} - b^{k+3}}{a - b} + \frac{b^{k+3} - c^{k+3}}{b - c} &+ \frac{c^{k+3} - a^{k+3}}{c - a} \\ &= \frac{a^{k+2} - b^{k+2}}{a - b} + \frac{b^{k+2} - c^{k+2}}{b - c} + \frac{c^{k+2} - a^{k+2}}{c - a} \\ &\quad + \frac{a^{k+1} - b^{k+1}}{a - b} + \frac{b^{k+1} - c^{k+1}}{b - c} + \frac{c^{k+1} - a^{k+1}}{c - a} \\ &\quad + \frac{a^k - b^k}{a - b} + \frac{b^k - c^k}{b - c} + \frac{c^k - a^k}{c - a}\end{aligned}$$

is a sum of three integers, and this completes the proof by induction.

Q.962 Prove that

$$\frac{1 \times 3 \times 5 \times \ldots \times 99}{2 \times 4 \times 6 \ldots \times 100} = (1 - \frac{1}{2})(1 - \frac{1}{4}) \ldots (1 - \frac{1}{100}) < \frac{1}{12}$$

[Hint: Try to discover a general result which can be proved by induction.]

ANS. First note that since $\frac{9}{10} < \frac{10}{11}$, $\frac{11}{12} < \frac{12}{13}$, and so forth, we have

$$\begin{aligned}\left(\frac{9 \times 11 \times \cdots \times 99}{10 \times 12 \times \cdots \times 100}\right)^2 &= \frac{9}{10} \times \frac{9}{10} \times \frac{11}{12} \times \frac{11}{12} \times \cdots \times \frac{99}{100} \times \frac{99}{100} \\ &< \frac{9}{10} \times \frac{10}{11} \times \frac{11}{12} \times \frac{12}{13} \times \cdots \times \frac{99}{100} \times \frac{100}{101} \\ &= \frac{9}{101}\end{aligned}$$

and so

$$\frac{9 \times 11 \times \cdots \times 99}{10 \times 12 \times \cdots \times 100} < \sqrt{\frac{9}{101}} < \frac{3}{10}\,.$$

Therefore

$$\frac{1 \times 3 \times 5 \times \cdots \times 99}{2 \times 4 \times 6 \times \cdots \times 100} < \frac{1}{2} \times \frac{3}{4} \times \frac{5}{6} \times \frac{7}{8} \times \frac{3}{10} = \frac{21}{256} < \frac{21}{252} = \frac{1}{12}\,.$$

Q.963 Starting from home I drive a certain distance due East. Then I turn 90° left and drive another distance, then turn left again and so on until I have driven $4n$ segments. Suppose the lengths of these segments are $1, 2, 3, \ldots, 4n$ kilometres but not necessarily in that order.

(a) What is the greatest distance I can eventually be from home?

(b) Can I end up at home?

ANS.

(a) Clearly the greatest distance will be obtained if we choose two directions, say north and east, and always go as far as possible in these directions and as little as possible in the others. So if we let N, S, E, W be the total distance travelled (in the positive sense) in each of the four directions, we will want

$$N{+}E = (2n{+}1){+}(2n{+}2){+}\cdots{+}4n = n(6n{+}1) \quad \text{and} \quad S{+}W = 1{+}2{+}\cdots{+}2n = n(2n{+}1)\,.$$

Now let V be the overall distance travelled north and H the overall distance travelled east, that is, $V = N - S$, $H = E - W$. Then

$$V + H = (N + E) - (S + W) = 4n^2$$

and the square of the total distance travelled is

$$V^2 + H^2 = V^2 + (4n^2 - V)^2 = 2V^2 - 8n^2V + 16n^4\,.$$

The graph of this expression, considered as a function of V, is a parabola opening upwards; therefore there will be no maximum turning point, and the expression attains its maximum value for either the maximum or minimum value of V. The minimum value of V is

$$V = N - S = \big((2n+1)+(2n+2)+\cdots+3n\big) - \big((n+1)+(n+2)+\cdots+2n\big) = n^2\,,$$

so $H = 3n^2$ and $V^2 + H^2 = 10n^4$; the maximum value of V gives the same result, as you may check for yourself. Hence the maximum possible distance from home is

$$\sqrt{V^2+H^2} = \sqrt{10}\,n^2\,.$$

(b) If $n = 1$ it is easy to see by trial and error that it is impossible to end up at home. If $n = 2$ we may travel along eight segments of distances

$$1, 2, 3, 4, 8, 7, 6, 5\,,$$

in that order; then the overall distance travelled east is $1 - 3 + 8 - 6 = 0$, and the overall distance north is likewise zero. If $n = 3$ the sequence

$$1, 2, 3, 4, 8, 6, 7, 5, 12, 10, 11, 9$$

achieves the same object. If we know how to get home after $4n$ segments then, having done so, we may continue with segments of lengths

$$4n+1\,,\ 4n+2\,,\ 4n+3\,,\ 4n+4\,,\ 4n+8\,,\ 4n+7\,,\ 4n+6\,,\ 4n+5\,,$$

which add no extra overall distance in either the vertical or horizontal directions and therefore return us home after $4(n+2)$ segments. Hence, we may always end up at home, provided n is greater than 1.

Q.964 The Fibonacci numbers are defined by

$$F_{n+1} = F_n + F_{n-1} \quad \text{where}\, n \geq 1$$

subject to $F_0 = F_1 = 1$. Show that

$$F_n = {}^{n}C_0 + {}^{n-1}C_1 + {}^{n-2}C_2 + \ldots + {}^{n-k}C_k$$

where $k = \frac{n}{2}$ if n is even, $(n-1)/2$ if n is odd.

Interpret this equation in Pascal's triangle.

ANS. Let A_n be the number of ordered lists of 1s and 2s such that the sum of all the numbers in the list is n. For example, $A_4 = 5$ since

$$1111\,,\ 112\,,\ 121\,,\ 211 \text{ and } 22$$

are the only possible lists adding up to 4. We shall use two different arguments to show that A_n is equal to the left hand side and the right hand side of the given equation; the two sides must therefore be equal to each other.

Firstly, $A_0 = 1$ since the "empty list" containing no numbers at all is the only list of 1s and 2s adding up to 0, and $A_1 = 1$ since the list containing just a single 1 is the only possibility adding up to 1. For $n \geq 1$, a list of 1s and 2s adding up to $n+1$ must consist of

- a 1, followed by a list adding up to n; or
- a 2, followed by a list adding up to $n-1$.

The number of possibilities is A_n in the first case, A_{n-1} in the second, and so

$$A_{n+1} = A_n + A_{n-1} \, .$$

Since the numbers A_n satisfy the same recurrence relation and the same initial conditions as the Fibonacci numbers, we have $A_n = F_n$ for all $n \geq 0$.

On the other hand, let us count the lists containing m twos and adding up to n. If there are m twos there must be $n - 2m$ ones and hence $n - m$ numbers in the list altogether; we can specify an ordered list by choosing which m places of the $n - m$ available are to be occupied by the 2s. This can be done in ${}^{n-m}C_m$ ways. The number of 2s in the list can be anywhere from a minimum of zero up to a maximum of $\frac{n}{2}$ if n is even, $\frac{n-1}{2}$ if n is odd. Thus

$$A_n = {}^nC_0 + {}^{n-1}C_1 + {}^{n-2}C_2 + \cdots + {}^{n-k}C_k \, ,$$

where k is as specified in the question. As explained above, this completes the proof.

This shows that in Pascal's triangle the "oblique diagonals" such as $1, 5, 6, 1$ (enclosed in boxes below) add up to the Fibonacci numbers.

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & \boxed{1} & & & \\
 & & 1 & & 4 & & \boxed{6} & & 4 & & 1 & & \\
 & 1 & & \boxed{5} & & 10 & & 10 & & 5 & & 1 & \\
\boxed{1} & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}$$

Q.965 A fraction is called a unit fraction if it has numerator 1.

(a) Show that every positive fraction can be written as a sum of distinct unit fractions.

(b) Given any two positive integers a and b, show that there exists a positive integer k such that ak can be written as a sum of distinct factors of bk.

ANS.

(a) First consider a positive fraction $\dfrac{a}{b} < 1$. Then $\dfrac{a}{b}$ must lie between two unit fractions, say

$$\frac{1}{n} \leq \frac{a}{b} < \frac{1}{n-1} \, ,$$

where clearly $n \geq 2$. So we can write

$$\frac{a}{b} = \frac{1}{n} + \frac{na-b}{nb} \,, \tag{$*$}$$

writing $\frac{a}{b}$ as a unit fraction plus a remainder. We can then repeat the procedure with the remainder until we have found the original fraction as a sum of unit fractions. Note that

$$\frac{a}{b} < \frac{1}{n-1} \,, \quad \text{so} \quad (n-1)a < b \,, \quad \text{so} \quad na - b < a \,;$$

thus in $(*)$ the numerator of the remainder is less than the numerator of the original fraction. Since these numerators cannot decrease indefinitely, the procedure must eventually terminate with a representation of $\frac{a}{b}$ solely in terms of unit fractions. Moreover,

$$\frac{na-b}{nb} = \frac{a}{b} - \frac{1}{n} < \frac{1}{n-1} - \frac{1}{n} = \frac{1}{n(n-1)} \leq \frac{1}{n}$$

since $n \geq 2$. Hence when the remainder in $(*)$ is written as a sum of unit fractions, all the denominators of these fractions must be greater than n, and this shows that the unit fractions we eventually find adding up to $\frac{a}{b}$ are all distinct.

If $\frac{a}{b} > 1$ then we can find an integer n such that

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \leq \frac{a}{b} < \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \frac{1}{n+1} \,.$$

We then have

$$\frac{a}{b} = \left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) + \frac{c}{d} \,, \tag{$+$}$$

where

$$\frac{c}{d} < \frac{1}{n+1} < 1 \,;$$

by the first part of the argument, $\frac{c}{d}$ can be written as a sum of distinct unit fractions with denominators greater than $n+1$, and substituting this sum into $(+)$ gives the required sum for $\frac{a}{b}$.

(b) Find a sum of distinct unit fractions

$$\frac{a}{b} = \frac{1}{n_1} + \frac{1}{n_2} + \cdots + \frac{1}{n_s}$$

as in (a), and let k be a common denominator for the fractions on the right hand side. Then

$$ak = \frac{bk}{n_1} + \frac{bk}{n_2} + \cdots + \frac{bk}{n_s}$$

is an expression of the required form.

Carlos Alberto da Silva Victor of Rio de Janeiro, Brazil, sent solutions to problems 957, 958, 959, 960, 961 and a partial solution to 962.

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NOT SO OBVIOUS (page 6)

$$\begin{array}{rr} & 29786 \\ + & 850 \\ + & 850 \\ \hline & 31486 \\ \hline \end{array}$$

STRIP PATTERN PUZZLE (page 11)

It's a deep frieze!